Math 145, Spring 2001 **Name:** ________________________

Exam 1

This test consists of 6 questions on 6 pages, totalling 100 points. You may use anything we have covered in the class notes and the class textbook freely, but you cannot receive help from other people. More specifically, unless otherwise stated, you may take as given anything that has been proven in class, in the homework, or in the reading.

1. (16 points) Let $X = \{(x, y) \mid y > 0\}$, and let the transformation $f : X \to X$ be defined by

$$f(x, y) = \left(-\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right).$$

Using our standard analytic methods, prove that f is conformal, or in other words, that f preserves angles (Euclidean or hyperbolic).

2. Let $\mathbf{v} = (2,3)$, let $C = (6,2)$, let $D = (2,7)$, and let m be the line with equation $2x + y = 7$.

(a) (8 points) As precisely as possible, describe the Euclidean isometry $R_{C,-\frac{\pi}{2}} \circ \rho_m$ as a particular reflection, rotation, translation, or glide-reflection, and prove that your answer is correct.

(b) (8 points) As precisely as possible, describe the Euclidean isometry $R_{D,\frac{\pi}{2}} \circ \tau_{\mathbf{v}} \circ R_{D,-\frac{\pi}{2}}$ as a particular reflection, rotation, translation, or glide-reflection, and prove that your answer is correct.

3. (18 points) In Euclidean geometry, prove that if ρ is a reflection, and γ is a glide-reflection (but not just a reflection), then $\rho\gamma$ is either a rotation or a translation. Furthermore, give a geometric condition describing exactly when $\rho\gamma$ is a rotation and when $\rho\gamma$ is a translation.

4. (16 points) Let $A = (1, 2)$, $B = (3, 2)$, and $C = (7, 2)$ be points in $\mathbf{H}^2$. Find a hyperbolic isometry f such that $f(A) = (0, k)$, $f(B) = (s, t)$, and $f(C) = (0, 1)$, with $k > 1$ and $s > 0$. (Do not find a formula for f; instead, describe f as a composition of hyperbolic reflections and Euclidean translations.)

5. For $a, b > 0$, let $R(a, b)$ be the **Euclidean** rectangle with corners $(0, 2)$, $(a, 2)$, $(0, 2+b)$, $(a, 2+b)$.

(a) (6 points) Give a qualitative description of what R looks like to a resident of the hyperbolic plane. Specifically, which of its sides are hyperbolically straight or hyperbolically curved? If curved, in which direction?

(b) (10 points) For a given $a, b > 0$, let $\mathrm{ha}(R(a, b))$ denote the hyperbolic area of $R(a, b)$, let $L(a, b)$ be the hyperbolic length of the left side of R, and let $B(a, b)$ be the hyperbolic length of the bottom side of R. Prove that

$$\lim_{a,b \to 0+} \frac{L(a, b)B(a, b)}{\mathrm{ha}(R(a, b))} = 1.$$

6. (18 points) (Class notes, Thm. 3.3.4) Suppose A, B, and C (resp. A', B', and C') are noncollinear points in $\mathbf{H}^2$ with $h(A,B) = h(A',B')$, $h(B,C) = h(B',C')$, and $h(A,C) = h(A',C')$. Prove that there exists a composition f of no more than three hyperbolic reflections such that $f(A) = A'$, $f(B) = B'$, and $f(C) = C'$.