

**Sample Exam 3**  
**Math 42, Fall 2014**

This is most of the second exam I gave in Math 42 in Spring 2014. Our exam may be more difficult than this one, but it should at least be comparable, and cover similar (though not the same) material. Exceptions: Our exam will not cover 10.3, but it will cover 8.8–8.9. Also, Ch. 10 is not represented on this sample, and you should expect something from Ch. 10.

1. (10 points) Thanks to the Heartbleed bug, everyone at Innitech has to choose a new password. The new passwords must consist of 6 capital letters (A–Z) followed by 7 digits (0–9). Furthermore, you are not allowed to have a password where all six chosen letters are the same (e.g., “QQQQQQ1234567” is not acceptable). How many possible passwords are there?

Show all your work, and briefly **JUSTIFY** your answer. In particular, you should not just have numbers; you should give some indication of the method you are using, and how it applies here.

2. (10 points) The Milpitas base of the secret agency SHELLED has been infiltrated both by the NSA and also by the evil organization HYDRANT. When the dust clears, the 46 total personnel remaining at the base include 22 members of NSA, 32 members of SHELLED, and 27 members of HYDRANT. To complicate matters, there are 13 people who are members of both NSA and SHELLED, 12 in both NSA and HYDRANT, and 15 in both SHELLED and HYDRANT. How many people at the base are in all three organizations?

Show all your work, and briefly **JUSTIFY** your answer. In particular, you should not just have numbers; you should give some indication of the method you are using, and how it applies here.

3. (10 points) The Giants have 34 remaining (identical) shares of their championship money to distribute among 19 non-player employees. Shares can only be distributed in whole numbers (i.e., no fractional/partial shares), and there are no limits on the number of shares an employee can receive (e.g., a given employee might receive no shares, or might receive all of them). How many different ways are there for the Giants to distribute the shares?

Show all your work, and briefly **JUSTIFY** your answer. In particular, you should not just have numbers; you should give some indication of the method you are using, and how it applies here.

4. (10 points) Compute  $\sum_{j=0}^k \binom{k}{j} 3^j 7^{k-j}$ . Show all your work, and briefly **JUSTIFY** your answer.

5. (12 points) Matilda has 31 llamas.

- (a) She first hands out 17 identical llama treats, making sure that no llama receives more than one treat. In how many ways can she do this?
- (b) She then puts 5 distinct decorative collars on llamas, again making sure that no llama receives more than one collar. In how many ways can she do this?

For each part, show all your work, and briefly **JUSTIFY** your answer. In particular, you should not just have numbers; you should give some indication of the method you are using, and how it applies here.

6. (12 points) Consider a sequence  $a_n$  that begins  $a_1 = 3$ ,  $a_2 = 7$ , 13, 21, 31, 43,  $\dots$ . Assuming this pattern continues, find a closed form for  $a_n$ . Show all your work.

7. (12 points) Give a **combinatorial proof** that, for positive integers  $n \geq k$ ,

$$\binom{n}{k} \binom{k}{2} = \binom{n}{2} \binom{n-2}{k-2}.$$

(A numerical proof using factorial formulas will receive partial credit.)

8. (14 points) Let  $b_n$  be the sequence defined by

$$\begin{aligned} b_1 &= 1, \\ b_2 &= 2, \\ b_m + 2b_{m+1} &= b_{m+2}, \end{aligned} \qquad \text{for } m \geq 1.$$

Use induction to prove that

$$2b_1 + 2b_3 + \dots + 2b_{2n-1} = \sum_{j=1}^n 2b_{2j-1} = b_{2n}.$$

Suggestion: Imitate the similar HW problem about the Fibonacci sequence. If you are not sure how to prove the identity, at least set up the induction.