Math 42
How to outline an induction proof

Suppose we want to do an induction proof of something like the following:

- Consider the function $f(b, n)$ defined by the following recursive algorithm:
  - If $n = 0$, output the value $f(b, n) = 1$.
  - If n is odd, output the value $f(b, n) = b * f\left(b^2, \lfloor n/2 \rfloor\right)$, where $*$ is multiplication and $\lfloor x \rfloor$ is the greatest integer $\leq x$.
  - Otherwise, output the value $f(b, n) = f(b^2, n/2)$.

  Prove that if n is an integer and $n \geq 0$, then $f(b, n) = b^n$.

To outline a proof of a theorem Thm(n) by (standard) induction on n, write the following:

- **Base case:** Verify that Thm(n) works for the smallest case of n.
- **Induction:** We would like to prove that Thm(k) implies Thm($k + 1$). So set up the following Assume/Conclude structure:
  - **Assume:** (Induction hypothesis) Thm(k) works (write out what this means).
  - Leave a hopeful blank space in the middle.
  - **Conclude:** (Induction step) Thm($k + 1$) works (write out what this means).

Then to finish the proof, fill in the blank space with an explanation of why assuming Thm(k) leads logically to conclude Thm($k + 1$).

Note that for strong induction, the induction hypothesis becomes:

> **Assume:** (Induction hypothesis) For (base case) $\leq n \leq k$, Thm(n) works (write out what this means).

For example, if the base case is $n = 1$, the strong induction hypothesis becomes:

> **Assume:** Thm(1) works, Thm(2) works, ..., Thm($k-1$) works, and Thm(k) works.

Examples. An induction outline is relatively straightforward for more mechanical proofs (e.g., proving equations involving summations), so we give two less mechanical examples.

Example 1: Suppose we want to prove:

> **Theorem:** For $n \geq 1$, the complete graph K_n has $\dfrac{n(n-1)}{2}$ edges.

The setup (standard induction) is:

- **Base case:** For $n = 1$, K_1 has 0 edges, and $\dfrac{1(0)}{2} = 0$, which checks.
- **Induction:**

  - **Assume:** The complete graph K_k has $\dfrac{k(k-1)}{2}$ edges.
  - Hopeful blank space:

    (???)

  - **Conclude:** The complete graph K_{k+1} has $\dfrac{(k+1)k}{2}$ edges.

Example 2: Here's the outline for the $f(b, n)$ problem mentioned above.

- **Base case:** For $n = 0$, $f(b, 0)$ returns $f(b, 0) = 1$, and $b^0 = 1$, so f returns the correct answer.

- **Induction step:**

  - **Assume:** For $0 \leq n \leq k$, Thm(n) works. In other words, for $0 \leq n \leq k$, $f(b, n) = b^n$.
  - Hopeful blank space:

    (???)

  - **Conclude:** Thm($k + 1$) works, i.e., $f(b, k + 1) = b^{k+1}$.