Math 42, Spring 2023 **Name:** ________________________
Exam 3

This test consists of 10 questions on 6 pages, totalling 100 points. You are not allowed to use books, notes, or calculators. Unless otherwise stated, you may take as given anything that has been proven in class, in the homework, or in the reading.

1. (5 points) Find the binary expansion of $(\mathrm{BF8})_{16}$ (i.e., convert hexadecimal to binary). No explanation necessary.

2. (6 points) Molly is eating w&w candies from a bag of hundreds of red, green, and brown w&w candies. What is the smallest number of candies she needs to take out of the bag to ensure she has at least 6 (six) of one color? Briefly **EXPLAIN** your answer.

3. (10 points) Find $\gcd(2^3 \cdot 3 \cdot 7^2, 2^4 \cdot 7^3 \cdot 11^2)$. Show all your work.

4. (10 points) How many strings of six uppercase English letters are there with no repeated letters and no Q or X? Leave your answer in the form of a sum or product, possibly involving binoimial coefficients, and briefly **EXPLAIN** your answer.

5. (10 points) At the 24-hour restaurant Wawful House, the All-Nighter Special includes your choice of four proteins (beef, pork, chicken, or tofu), a choice of 3 sauces from among a total of 7, and a choice of 2 side dishes from among a total of 9. How many possible Wawful House All-Nighter Specials are there? Leave your answer in the form of a sum or product, possibly involving binoimial coefficients, and briefly **EXPLAIN** your answer.

6. (8 points) A 12-hour clock currently shows 11:00. What time will that same clock show 77 hours from now? Show all your work.

7. (12 points) **PROOF SETUP** (i.e., not a full proof):

Suppose we want to prove the following statement for all positive integers n.

> **Theorem:** Every positive integer n is equal to a sum of the form
>
> $$n = a_0 10^0 + a_1 10^1 + a_2 10^2 + \cdots + a_d 10^d$$
>
> for some nonnegative integer d and some coefficients a_i, where $0 \leq a_i \leq 9$.

Set up a proof by **strong** induction as follows:

(a) Prove the basis case.

(b) For the inductive step, write out the assumption (the inductive hypothesis) and the conclusion and leave a space in the middle for the bulk of the argument.

Do **NOT** attempt to complete the proof.

8. (12 points) Recall that an ordinary deck of 52 cards has cards in 13 *ranks* (A, 2, 3, ..., Q, K) and 4 *suits* (clubs, diamonds, hearts, spades).

What is the probability that a five-card poker hand has three of a kind and nothing else? (I.e., three cards of the same rank, and two other cards, each of a different rank, like QQQ89.) Leave your answer in the form of a sum, product, or quotient, possibly involving binoimial coefficients, and briefly **EXPLAIN** your answer.

9. (12 points) If we randomly select a permutation of the 26 lowercase letters of the English alphabet:

(a) What is the probability that x immediately precedes y in that permutation?

(b) What is the probability that x and y are not next to each other in that permutation?

Leave each answer in the form of a sum, product, or quotient, possibly involving binoimial coefficients, and briefly **EXPLAIN** your answer.

10. (15 points) Use (regular) induction to prove: For any integer $n \geq 0$, we have that

$$3^0 + 3^1 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 1}{3 - 1}.$$

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