Final Exam

Math 42

Fall 2021

NAME:

Instructions:

1. You have 135 minutes to complete the exam.
2. You must show all of your work to receive full credit. Be sure to read all directions and provide explanations when requested.
3. There are 100 points on the exam. You should have 9 pages including the cover page.
4. Please write your solutions on the assigned area below each question.
5. No cell phones, calculators, notes, books, use of internet and computer are allowed at any point during the exam.
6. Good Luck!

1. (a) (5 points) Write the truth table for the proposition $p \vee (p \to \neg q)$.

(b) (3 points each) Let $S(x)$ be the statement "x knows how to ski." and $L(x)$ be the statement "x likes sports". Express the following statements as logical expressions using quantifiers and predicates. The domain is the set of all people in the world.

i. All people like sports.

ii. People who like sports know how to ski.

2. We are making passwords in such a way that each character in the password is a digit or an uppercase letter.

   (a) (4 points) How many such passwords of length n are there?

   (b) (4 points) How many passwords of length 10 start with a digit?

   (c) (5 points) What is the probability that a random password of length 20 contains **at least** one digit?

   (d) (5 points) What is the probability that a random password of length 20 contains **exactly** three digits?

3. (a) (5 points) Let A, B, C be subsets of the universal set U. Draw the Venn Diagram for $(A-B)-C$.

(b) (4 points) What is the range of the following function? Explain.

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = |x-3| + 5$$

(c) (5 points) Find the inverse of the following function or explain why it does not exist.

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = \frac{x-2}{5}$$

4. (a) (6 points) Find the integer a in $\{0, 1, \ldots, 26\}$ such that $a \equiv -15 \pmod{27}$. Explain.

(b) (6 points) Which positive integers less than 12 are relatively prime to 12?

5. (a) (4 points) List all elements in the relation $R = \{(a, b)|a, b \in \mathbb{Z}\ and\ ab = 2\}$.

(b) (6 points) Determine if R is reflexive, symmetric, or transitive. Explain.

6. (a) (5 points) Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3 \ldots$ if $a_n = 1 + (-1)^n$.

(b) (6 points) Find $f(2), f(3), f(4)$ if f is defined recursively by $f(0) = -1$, $f(1) = 2$, and for $n = 1, 2, 3, \ldots$ we have $f(n+1) = f(n-2) + 1$.

7. (a) (7 points) Prove that if x and y are rational numbers, then $x + y$ is also rational.

(b) (7 points) Use induction to prove that $1^2 + 3^2 + 5^2 + \ldots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ whenever n is a nonnegative integer.

8. (2 points each) Mark each statement true or false. No need for explanation.

   (a) The negation of $\forall x \exists y (P(x) \wedge Q(y))$ is $\forall y \exists x (\neg P(x) \vee \neg Q(y))$.

   (b) If $A \subseteq B$, then $|A| < |B|$.

   (c) $\{(x, y) |\, x, y \in \mathbb{Z}, xy = 0\}$ is a finite set.

   (d) If A, B, C are sets and $A \cap B = A \cap C$, then $B = C$.

   (e) In any group of 5 integers there are two with the same remainder when divided by 4.