How can we multiply vector times vector "like numbers"?

1-D: We can $+$, $-$, mult, divide ordinary real numbers.

2-D: We can $+$, $-$, mult complex numbers, using the rule $i^2 = -1$, and divide by taking reciprocals: $\dfrac{1}{z} = \dfrac{\overline{z}}{z\overline{z}}$, where, if $z = a + bi$, $\overline{z} = \overline{a + bi} = a - bi$, and

$$z\overline{z} = (a + bi)\overline{(a + bi)} = a^2 + b^2$$

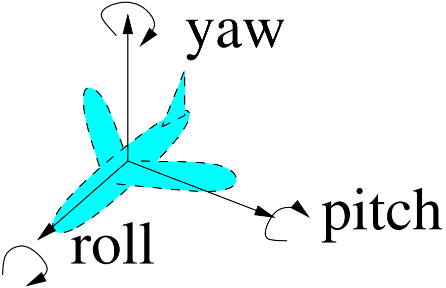
is the ordinary 2-D dot product of z with itself.

3-D: Today: cross-product. Does that work?

How about 4-D? Higher?

How can we avoid gimbal lock?

The space of all possible 3-D bearings (directions) of an aircraft is itself 3-dimensional: roll, pitch, and yaw.



Problem: The “RPY” coordinates for the space of all rotations have singularities, a real-life phenomenon known as *gimbal lock*. (Famously, Apollo 13 ran into trouble because of gimbal lock.)
Click here for a 30-second explanation of gimbal lock.

Cross-product FAIL

We can't use the cross-product to multiply 3-D vectors like numbers, because $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v}$, which means that no vector can behave like the number 1.
(cue the sad trombones)

Much less obviously: There is *no way* to define a multiplication in 3-D that makes 3-D vectors "act like numbers," with $+$, $-$, multiply, and divide, in a natural way.

How about 4-D?

Quaternions (Hamilton, 1843)

Call the standard unit vectors in $\mathbb{R}^4$ $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Then Hamilton's rules for multiplying vectors in $\mathbb{R}^4$ are:

$$\begin{array}{ccc} \mathbf{ij} = \mathbf{k} & \mathbf{jk} = \mathbf{i} & \mathbf{ki} = \mathbf{j} \\ \mathbf{ji} = -\mathbf{k} & \mathbf{kj} = -\mathbf{i} & \mathbf{ik} = -\mathbf{j} \\ \mathbf{i}^2 = -\mathbf{1} & \mathbf{j}^2 = -\mathbf{1} & \mathbf{k}^2 = -\mathbf{1} \end{array}$$

and $\mathbf{1}$ acts like the usual number 1. More succinctly,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1},$$

as Hamilton famously carved on Brougham Bridge in Dublin. Notably, however, we must pay the price that quaternion multiplication is no longer *commutative*.

Quaternions combine dot and cross products

If $\mathbf{v}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$, then in quaternion multiplication,

$$\mathbf{v}_1\mathbf{v}_2 = (-\mathbf{v}_1 \cdot \mathbf{v}_2)\mathbf{1} + (\mathbf{v}_1 \times \mathbf{v}_2),$$

where $\mathbf{v}_1 \cdot \mathbf{v}_2$ is the dot product and $\mathbf{v}_1 \times \mathbf{v}_2$ is the cross product written in $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ notation. (Try it out yourself!)

That's why the dot product is sometimes called an *inner product* and the cross product is occasionally called the *outer product*, after the inner and outer terms in quaternion multiplication.

Quaternions encode rotations

If $\mathbf{q} = r\mathbf{1} + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a quaternion, then the reciprocal of $\mathbf{q}$ is $\mathbf{q}^{-1} = \frac{1}{\mathbf{q} \cdot \mathbf{q}}\overline{\mathbf{q}}$, where $\overline{\mathbf{q}} = r\mathbf{1} - a\mathbf{i} - b\mathbf{j} - c\mathbf{k}$, just as in the complex numbers. (Again, try it yourself! And note that $\mathbf{q}\overline{\mathbf{q}} = r^2 + a^2 + b^2 + c^2$.)

It can be shown that for any nonzero $\mathbf{q}$ and $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the function $R_{\mathbf{q}}(\mathbf{x}) = \mathbf{q}\mathbf{x}\mathbf{q}^{-1}$ describes a 3-D rotation, and every rotation can be expressed in this way. In fact, since the magnitude of $\mathbf{q}$ does not affect $R_{\mathbf{q}}$, we can assume that $\mathbf{q} \cdot \mathbf{q} = 1$, i.e., that $\mathbf{q}$ lies on the (3-dimensional) unit sphere in $\mathbb{R}^4$. Thus, the 3-sphere gives coordinates for the space of all rotations that are better than RPY in many ways.

See the Wikipedia entry on quaternions and rotations.
See also this page describing a particular rotational application of quaternions.

What comes next in the sequence 1, 2, 4, 8?

Trick question! The answer is **nothing**. That is, we can only find a "nice" way to multiply vectors in dimensions 1, 2, 4, and 8 (!).

In 1-D, we have real numbers, which are ordered, commutative, and associative.

In 2-D, we have complex numbers, which are no longer ordered, but are still commutative and associative.

In 4-D, we have quaternions, which are no longer ordered or commutative, but are still associative. Everything else still works: $+$, $-$, mult, div (as we saw).

In 8-D, we have *octonions*, which are neither ordered, commutative, nor associative. Everything else still works.

And that's all! See the Wikipedia entry on Hurwitz's theorem.