Sample Final Exam
Math 221B, Spring 2022

This exam was a take-home *non-comprehensive* exam, so please do not use it as a guide for our comprehensive take-home final. The questions are good practice questions, though.

All fields are assumed to be characteristic 0.

1. (20 points) It is a fact that if G is a group of order p^n (p prime), there exists a normal subgroup N of G such that $[G : N] = p$.

Let K be a Galois extension of a field F such that $[K : F] = p^n$ (p prime). Prove that any $\alpha \in K$ is equal to a expression obtained by starting with elements of F and repeated applying field operations and taking radicals (kth roots).

2. Let K be the splitting field of $f(x) = x^5 - 2$ over $\mathbf{Q}$ and let $G = G(K/\mathbf{Q})$. Let $\alpha = \sqrt[5]{2}$, and let $\zeta = e^{2\pi i/5}$. In the following, you may find Thm. 12.4.9 to be useful.

(a) (10 points) Prove that $\zeta \in K$.

(b) (15 points) Prove that $[K : \mathbf{Q}] = 20$.

(c) (15 points) Let $H_1 = G(K/\mathbf{Q}(\alpha))$. Find a smallest possible set of explicit generators for H_1 in terms of permutations of the roots of $f(x)$. (Suggestion: Given $\sigma \in H_1$, what is $\sigma(\zeta\alpha)$?)

(d) (15 points) Let $H_2 = G(K/\mathbf{Q}(\zeta))$. Find a smallest possible set of explicit generators for H_2 in terms of permutations of the roots of $f(x)$. (Suggestion: Given $\sigma \in H_2$, what is $\sigma(\alpha)$?)

(e) (15 points) Is H_1 normal in G? Is H_2 normal in G? Prove your answer.

(f) (10 points) Prove that G contains a subgroup of order 10. What does that tell you about fields L such that $\mathbf{Q} \subseteq L \subseteq K$?

Aside: It turns out that the above calculations classify G by giving a presentation for G, but you do not need to prove that.