Sample Exam 3
Math 221B, Spring 2022

Note that this was a take-home exam, and therefore *way* too long for an in-class exam. The problem subject matter is representative, though.

1. (12 points) Write $u_1^3 + u_2^3 + u_3^3$ in terms of the elementary symmetric functions in 3 variables. Show all your work.

2. (14 points) Let $f(x)$ be a real quintic polynomial with exactly 3 real roots, and suppose that all of the roots of $f(x)$ are distinct. What can you say about the sign of the discriminant of $f(x)$? Prove your answer.

3. (14 points) Let $x^4 + ax^3 + bx^2 + cx + d$ be an irreducible polynomial over F, and let α be a root of f in an extension field K. Find a formula for α^{-1} as a polynomial in α, in terms of a, b, c, d, and justify your answer.

4. (14 points) Let F be a field of order 2^p, where p is prime, and define $\rho : F \to F$ by $\rho(x) = x^2$.

(a) Prove that ρ is an automorphism of F.
(b) Prove that $\rho(x) = x$ if and only if $x = 0, 1$.
(c) Prove that the order of ρ (as an automorphism of F) is p.

5. (14 points) Let F be a field and suppose we have $a, b \in F$ such that none of a, b, ab are squares in F. Let $K = F(\sqrt{a}, \sqrt{b})$. Prove carefully that $[K : F] = 4$.

6. (16 points) Let $K = \mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7})$. You may take it as given that $G(K/\mathbf{Q})$ is generated by $\tau_2, \tau_3, \tau_5, \tau_7$, where τ_a is the transposition $(\sqrt{a} \;\; -\sqrt{a})$.

Find the Galois group $G(K/L)$ for each of the following intermediate fields L. Justify each answer.

(a) $L = \mathbf{Q}(\sqrt{210})$.
(b) $L = \mathbf{Q}(\sqrt{2}, \sqrt{3})$.
(c) $L = \mathbf{Q}(\sqrt{6}, \sqrt{15}, \sqrt{35})$.

7. (16 points) Suppose K/F is a Galois extension such that $G(K/F) = S_3$. Prove that for some $a \in F$, K contains a subfield $F(\sqrt{a})$ that is not equal to F.