

Sample Exam 2
Math 221B, Spring 2022

1. (10 points) Give an example of an ordinary integer prime $p > 2$ (i.e., p is prime in \mathbf{Z}) that splits in $\mathbf{Z}[i]$, and give the factorization of your chosen p into Gaussian primes.
2. (15 points) Let R be the ring of algebraic integers of an imaginary quadratic number field.
 - (a) State what the Main Lemma says about a nonzero ideal A of R .
 - (b) Explain the importance of the Main Lemma in the construction of the class group of R . (If you don't remember why the Main Lemma is important here, at least define the class group.)
3. (15 points) Let $\delta = \sqrt{-5}$, and let A be the lattice of integer linear combinations of $\{4, 1 + \delta\}$. Is A an ideal of $\mathbf{Z}[-5]$ or not? Justify (prove) your answer.
4. (15 points) Let d be a square-free negative integer such that $d \equiv 2$ or $3 \pmod{4}$, let $\delta = \sqrt{d}$, and let R be the ring of integers of $\mathbf{Q}[\delta]$.
 - (a) Define what it means for 5 to ramify in R .
 - (b) Prove that if 5 divides d (and 25 does not divide d , since d is square-free), then 5 ramifies in R , and the ideal $P = (5, \delta)$ divides 5.
5. (15 points) Let $\delta = \sqrt{-26}$, and let A and B be ideals in $\mathbf{Z}[\delta]$ given by

$$A = (3, 1 + \delta) \qquad B = (9, 1 + \delta).$$

Prove that AB is equal to the principal ideal $(1 + \delta)$.

6. (15 points) Let a be an even integer, and let $f(x) = x^{10} - x^9 + ax + 2$. Prove that if $f(x) = g(x)h(x)$ in $\mathbf{Z}[x]$, and neither $g(x)$ nor $h(x)$ is a constant polynomial, then one of $g(x)$ or $h(x)$ must have degree 1. (Suggestion: Use reduction mod 2.)
7. (15 points) Let $\delta = \sqrt{-38}$ and let R be the ring of integers of $\mathbf{Q}[\delta]$. Note that

$$2\sqrt{\frac{38}{3}} \approx 7.118.$$

Determine the class group of R , using $N(2 + \delta)$ and $N(4 + \delta)$.