Sample Exam 1
Math 221B, Spring 2022

1. (14 points) Let R be a ring, and let I and J be ideals in R. Define the ideal IJ.

2. (13 points) **COUNTEREXAMPLE:** Give an example of a unique factorization domain R that is not a principal ideal domain. Explain how you know that R is a UFD, and give an example of a non-principal ideal in R.

3. (13 points) **COUNTEREXAMPLE:** Give an example of a ring R and a nonempty subset S of R such that S is closed under addition and subtraction, but S is *not* an ideal in R.

4. (15 points) **PROOF QUESTION.** Find a generator for the kernel of the homomorphism $\varphi : \mathbf{Q}[x] \to \mathbf{R}$ given by $\varphi(f(x)) = f(1-\sqrt{3})$ (i.e., φ sends x to $1-\sqrt{3}$), with proof.

5. (15 points) **PROOF QUESTION.** Identify the ring $\mathbf{Z}[x]/(x^2+3, x-4)$. Prove your answer.

6. (15 points) **PROOF QUESTION.** Let $R = \mathbf{Q}[x]/(x^4+3x^2+2)$.

(a) Prove that x^2+2 is an idempotent in R.

(b) Prove that R is isomorphic to the product of two rings. Give explicit descriptions of the two rings.

7. (15 points) **PROOF QUESTION.** Let a, b, d, δ all be nonzero elements of an integral domain R, and suppose that

- $(a) + (b) = (\delta)$;
- d divides both a and b; and
- $ax + by = d$ for some $x, y \in R$.

Prove that d divides δ and δ divides d.