Sample Exam 2
Math 221B, Spring 2020

1. (10 points) Give an example of an ordinary integer prime $p > 2$ (i.e., p is prime in $\mathbf{Z}$) that splits in $\mathbf{Z}[i]$, and give the factorization of your chosen p into Gaussian primes.

2. (15 points) Let R be the ring of algebraic integers of an imaginary quadratic number field.

(a) State what the Main Lemma says about a nonzero idea A of R.

(b) Explain the importance of the Main Lemma in the construction of the class group of R. (If you don't remember why the Main Lemma is important here, at least define the class group.)

3. (15 points) Let $\delta = \sqrt{-5}$, and let A be the lattice of integer linear combinations of $\{4, 1+\delta\}$. Is A an ideal of $\mathbf{Z}[-5]$ or not? Justify (prove) your answer.

4. (15 points) Let d be a square-free negative integer such that $d \equiv 2$ or $3 \pmod 4$, let $\delta = \sqrt{d}$, and let R be the ring of integers of $\mathbf{Q}[\delta]$.

(a) Define what it means for 5 to ramify in R.

(b) Prove that if 5 divides d (and 25 does not divide d, since d is square-free), then 5 ramifies in R, and the ideal $P = (5, \delta)$ divides 5.

5. (15 points) Let $\delta = \sqrt{-26}$, and let A and B be ideals in $\mathbf{Z}[\delta]$ given by

$$A = (3, 1+\delta) \qquad\qquad B = (9, 1+\delta).$$

Prove that AB is equal to the principal ideal $(1+\delta)$.

6. (15 points) Let a be an even integer, and let $f(x) = x^{10} - x^9 + ax + 2$. Prove that if $f(x) = g(x)h(x)$ in $\mathbf{Z}[x]$, and neither $g(x)$ nor $h(x)$ is a constant polynomial, then one of $g(x)$ or $h(x)$ must have degree 1. (Suggestion: Use reduction mod 2.)

7. (15 points) Let $\delta = \sqrt{-38}$ and let R be the ring of integers of $\mathbf{Q}[\delta]$. Note that

$$2\sqrt{\frac{38}{3}} \approx 7.118.$$

Determine the class group of R, using $N(2+\delta)$ and $N(4+\delta)$.