Math 221A, problem set 01
Due: Mon Feb 06
Last revision due: Mon Apr 03

Problems to be turned in: Problem x.y.z of Artin denotes problem y.z in Chapter x.

1. Let R be the set $\mathbf{R} \cup \{\infty\}$ (i.e., the set of all real numbers along with the symbol ∞), with operations $\oplus$ and $\otimes$ given by

$$a \oplus b = \min(a, b),$$
$$a \otimes b = a + b,$$

   for all $a, b \in R$, where we define $\min(a, \infty) = \min(\infty, a) = a$ and $a + \infty = \infty + a = \infty$ for all $a \in R$.

   It turns out that R is almost a ring, but not quite. For each of the axioms of a ring that R does **not** satisfy, use a counterexample to show that R fails to satisfy that axiom. (Treat the pieces of compound axioms separately; e.g., for the axiom of R being an abelian group, treat associativity, commutativity, identity, and additive inverse separately.)

2. Artin 11.1.6(b).

3. Let $\mathbf{Z}\left[\frac{1}{2}\right]$ be the smallest subring of $\mathbf{Q}$ that contains both $\mathbf{Z}$ and $\frac{1}{2}$.
   (a) Describe all reduced fractions k/n ($k, n \in \mathbf{Z}$) in $\mathbf{Z}\left[\frac{1}{2}\right]$. (No proof necessary.)
   (b) Which elements of $\mathbf{Z}\left[\frac{1}{2}\right]$ are units? Prove your answer.

4. Artin 11.3.2.

5. Let $\varphi : \mathbf{Z}[x] \to \mathbf{C}$ be defined by $\varphi(f(x)) = f(1 + \sqrt{3}i)$. Find a generator or generators for $\ker \varphi$.

6. Determine all ideals of the ring $\mathbf{Z}\left[\frac{1}{2}\right]$ from problem 3, with proof.

7. Let I and J be ideals of a ring R.
   (a) Let $A = \{xy \mid x \in I, y \in J\}$. Give an example of a ring R and ideals I, J where A is **not** an ideal of R.
   (b) Now let

$$IJ = \left\{ \sum x_n y_n \mid x_n \in I, y_n \in J \right\}.$$

   In other words, let IJ be the set of all *finite sums* of products of elements of I and J. Explain how IJ relates to $I \cap J$.
   (c) Prove that IJ is an idea of R.