When you're connected by zoom:

- Use a laptop or desktop with a large screen so you can read these words clearly.
- To conserve bandwidth, please turn off your camera.
- Please mute your microphone unless I call on you.
- Please have the chat window open to ask questions.
- Take-home final due in 1 week. (But all deadlines are elastic.)
- Today's DJ: Trent.

Take-home final out by Wed.

Kurosh Subgroup Theorem

Theorem (Kurosh)

Every subgroup of a free group is free.

For simplicity, we discuss and prove the finite index case, though everything works in general.

The fundamental group of a basepointed X-labelled graph

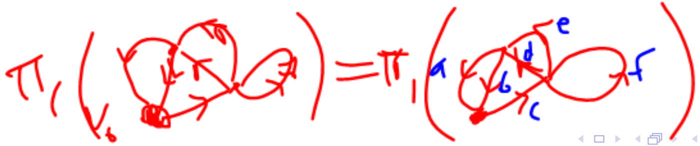
Let Γ be an X-labelled graph with basepoint v_0.

Definition

The **fundamental group of** Γ, written $\pi_1(\Gamma, v_0)$, is:

- Elements: Reduced loops starting at v_0.
- Operation: Concatenation.
- Identity: Empty word 1.
- Inverse of a loop: The same loop, traversed backwards.

Observation: If we change the labels on Γ, $\pi_1(\Gamma, v_0)$ doesn't change; we just have different names for the same paths.

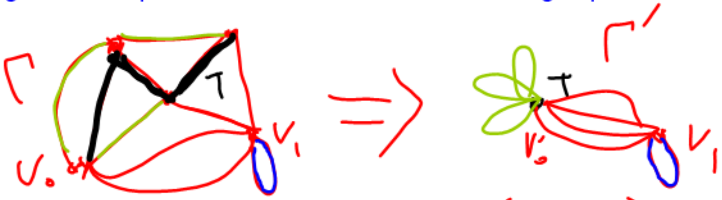


The Retract Lemma

Theorem

Let Γ be a basepointed X-labelled graph, let T be a tree contained in Γ, and let Γ' be the basepointed labelled graph obtained by replacing T with a single vertex and giving each edge a different label in some alphabet Y. Then $\pi_1(\Gamma, v_0) = \pi_1(\Gamma', v_0')$.

Collapsing a tree to a point does not affect the fundamental group.



$$\pi_1(\Gamma, v_0) = \pi_1(\Gamma', v_0')$$

Proof of the Retract Lemma

and concatenations map to concatenations

First: Relabel Γ so that every edge has a different label.
Loops map to loops, so enough to show that:

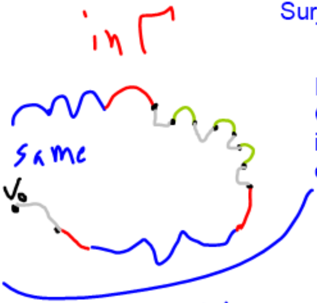
- (Surjective) Every loop in Γ' is the image of some loop in Γ.
- (Injective) Nontrivial reduced loops map to nontrivial reduced ^in Γ loops. in Γ'

— outside T
— inside T
— T, not T

Surjectivity: Suppose we have a loop in Gamma'. Illustrate using example.

Red-green-red paths may pull back to disconnected pieces; similarly, a path that ends in a red edge in Gamma' might not return to basepoint when pulled back.

But remember, any two points in a tree T are connected by a unique path within T. So we can fill all those gaps with paths from within T to get a loop starting and ending at v_0:



Surjectivity follows.

Injectivity: Suppose we have a reduced loop in Gamma. How could this map to a non-reduced loop in Gamma'? Look at all possible cases of consecutive edges in the loop in Gamma':

$\Gamma \xleftarrow{\text{from}} \Gamma'$

Blue-blue

These edges inject, so can't cancel

Blue-red

different color edges so different labels

Same argument works for green-red.

(this is mapped into Gamma')

from

Green-green

If this happened w/reduced black path, then we would lose injectivity.

Fortunately, there is a unique reduced path between any two points in a tree, including the case where the two points are the same. So the picture in Gamma is actually one where the black path is trivial:

So actually, we didn't have a reduced path in Gamma; contradiction. Same argument holds for red-red cancellation.

One case left: What if we have a reduced loop that's entirely black? Well, there's only one such path, namely, the trivial path. So the only element of the kernel of our homomorphism is the trivial path.

Fundamental groups of graphs are free

Corollary

Let Γ be a finite basepointed X-labelled graph with basepoint v_0, E edges, and V vertices. Then $\pi_1(\Gamma, v_0) = F_n$, where $n = E - V + 1$.

Proof: Given Γ with V vertices and E edges:

(V=1, done)

Lose one edge, one vertex, so E-V+1 remains constant. (Actually V-E is Euler) Result follows by induction on V.

Same proof works for infinite graphs, but you have to prove things about spanning trees of infinite graphs.

Covering spaces

$\to a$

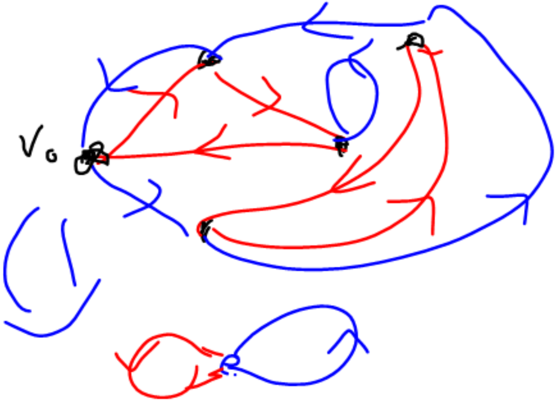
$\to b$ $\quad n=2$

Let $X = \{a_1, \ldots, a_n\}$.

Definition

We call a completely X-labelled graph a **covering space** of the n-leaved rose:

graph map
induced by
colors & arrows



Coset diagrams

Definition

$n=2$

For $H \leq F_n$, we define the (right) **coset diagram** of H to be the basepointed X-labelled graph given by:

For any $g \in F_n$, $a \in X$

$$Hg \xrightarrow{a} Hga$$

$$Hg \longrightarrow Hgb$$

Gives completely X-labelled graph whose vertices are the right cosets of H.

Basept: H

Main Theorem of covering spaces

Let $X = \{a_1, \ldots, a_n\}$.

Theorem

There exists a bijection between subgroups of F_n and basepointed covering spaces of the n-leaved rose, given by:

- **Subgroup to covering space:** *Map H to its basepointed coset diagram.*
- **Covering space to subgroup:** *Map (Γ, v_0) to $\pi_1(\Gamma, v_0)$.*

~~**Proof:**~~

$$H \longrightarrow \text{coset diag basept } H$$

Check: These maps are inverses.

$$\pi_1(\Gamma, v_0) \longleftarrow (\Gamma, v_0)$$

Proof of the Kurosh Subgroup Theorem

H subgroup of finite index in F_n.
By Correspondence of Covering Spaces, H is isomorphic to fundamental group of a graph, which we showed must be a free group.

Residual finiteness of free groups

Here's a slightly fancier result about free groups.

Theorem

Given $1 \neq w \in F_n$, there exists a finite quotient $\rho : F_n \to G$ such that $\rho(w) \neq 1$.

We say that free groups are **residually finite**: Given a nontrivial element w of a free group F_n, there exists a finite quotient of F_n in which some nontrivial "residue" survives.

Reducing the problem

Enough to show that:

Theorem

Given $1 \neq w \in F_n$, there exists a a subgroup H of finite index in F_n such that $w \notin H$.

For in that case, let N be the intersection of the finitely many conjugates of H. This is a subgroup of finite index in F_n, and $\rho : F_n \to F_n/N$ is the desired finite quotient.

Proof by example

In $F_2 = \langle a, b \rangle$, take $w =$