When you're connected by zoom:

- Use a laptop or desktop with a large screen so you can read these words clearly.
- To conserve bandwidth, please turn off your camera.
- Please mute your microphone unless I call on you.
- Please have the chat window open to ask questions.
- Take-home exam 3 due Wed. (But all deadlines are elastic.)
- Today's DJ: Ravi. PS11: Due Mon Problem session Fri 10:30

Galois theory in a nutshell

Suppose K/F is the splitting field of irred $f \in F[x]$.

- **Galois theory** relates the field theory properties of K/F (e.g., degree) and the group-theoretic properties of $G = G(K/F)$.
- G permutes roots $\alpha_1, \ldots, \alpha_n$ of f and is isomorphic to a transitive subgroup of S_n.
- Main Theorem: There is a bijective, inclusion-reversing correspondence between $H \leq G$ and $F \subseteq L \subseteq K$ sending H to fixed field K^H and L to $G(K/L) \leq G$. Moreover, L/F is Galois iff $H = G(K/L) \lhd G$, in which case $G(L/F) = G/H$.
- Typical problem: Given $f(x)$, compute $G(K/F)$.
- Big question: When can $\alpha_1, \ldots, \alpha_n$ be written as a function of the coefficients of $f(x)$ in an expression using only kth roots (generalizing quadatic formula)? If so, $f(x) = 0$ is **solvable by radicals**.
- Big answer: $f(x)$ solvable by radicals iff G solvable group.

Intersections between Analysis and Algebra:

* Fourier analysis + number theory = analytic number theory (Goldston)
See: Riemann Hypothesis, proof of Fermat's Last Theorem
(Schettler)

Langlands program

* Can also do Fourier analysis on groups
Possible thesis topic: Can you hear the shape of a group? (Fourier analysis on finitely presented groups)
(Me - learn more Fourier analysis)

* Analysis-based algebraic topology (L^2 homology/cohomology)
(Me - Non-commutative ring theory)

* Using class field theory to create examples in measure theory
(Hambrook)

Geometric group theory

Geometric group theory is the study of groups by treating them as geometric objects. Typical group to study: Free group. Let $X = \{a_1, \ldots, a_n\}$ be an **alphabet**.

$abcc^{-1}ba^{-1} = abba^{-1} = ab^2a^{-1}$

Definition

A **word** in X is something like $a_1 a_3 a_2^{-1} a_3^{-1} a_2$.
A **reduced word** in X is a word without $a_i a_i^{-1}$ or $a_i^{-1} a_i$.

Definition

F_n, the **free group on** n **generators**, is given by:

- Elements: Reduced words in X. $= \{a_1 \ldots a_n\}$
- Operation: Concatenation with reduction.
- Identity: Empty word (written as 1).
- Inverse: Socks-and-shoes. $(abc)^{-1} = c^{-1}b^{-1}a^{-1}$

Skip: Important but boring part where we check this all works.

Examples of operation, inverse:

$(abc)(c^{-1}b^{-1}a^{-1}) = abcc^{-1}b^{-1}a^{-1} = abb^{-1}a^{-1} = aa^{-1} = 1$

(Worry: What if you reduce differently?)

Kurosh Subgroup Theorem

Theorem (Kurosh)

Every subgroup of a free group is free.

For simplicity, we discuss and prove the finite index case, though everything works in general.

Now with cartoons!

Book project: "Geometric group theory: the comic book"

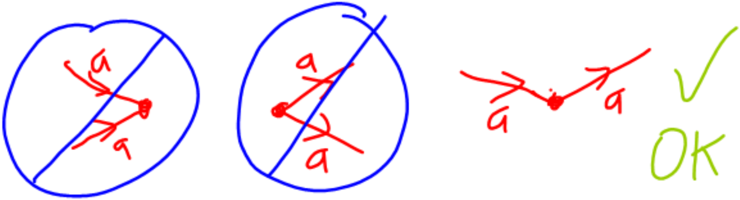
X-labelled graphs

$= \{a, b, c, \ldots\}$

Let X be an alphabet. We assume all graphs are connected and directed.

Definition

An X-**labelled graph** is a directed graph Γ whose edges are labelled with elements of X such that no two edges coming into (resp. out of) a vertex are labelled with the same letter.



Definition

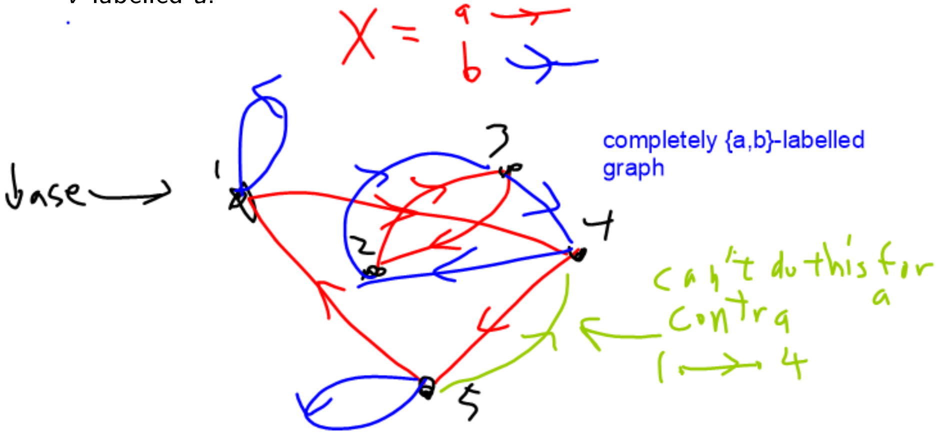
A **basepointed** X-labelled graph is an X-labelled graph Γ with a chosen vertex $v_0 \in \Gamma$, called the **basepoint** of Γ.

"basepoints = conjugation": See permutation gps, normal Main Thm Galois

Completely X-labelled graphs

Definition

A **completely X-labelled graph** is an X-labelled graph Γ such that for every vertex $v \in \Gamma$ and every $a \in X$, there is exactly one edge coming into v labelled a and exactly one edge coming out of v labelled a.



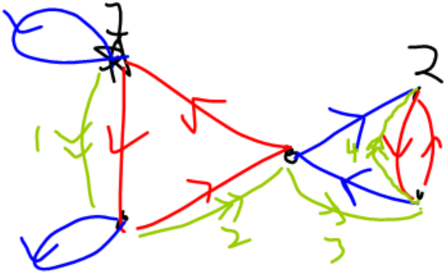
Paths in an X-labelled graph

Let Γ be an X-labelled graph.

Definition

(If Gamma is not completely labelled)

A **path** in Γ is specified by giving a starting vertex and a word in X. (Not every such description corresponds to a path, though.)



(Most of previous example)

If you start at 2, b doesn't give a path.

Start at 1: a a b^{-1} a^{-1} gives indicated green path.

Definition

Loop at 1: a a b^{-1} a^{-1} b^{-1} a.

A **loop** in Γ is a path that starts and ends at the same vertex.

The fundamental group of a basepointed X-labelled graph

Let Γ be an X-labelled graph with basepoint v_0.

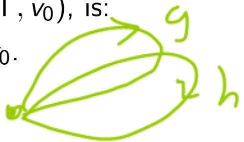
Reduced path: No backtrack (no a a^{-1})

Definition

(with reduction)

The **fundamental group of** Γ, written $\pi_1(\Gamma, v_0)$, is:

- Elements: Reduced loops starting at v_0.
- Operation: Concatenation.
- Identity: Empty word 1. (= empty loop)
- Inverse of a loop: The same loop, traversed backwards.



Example: F_3 is the fundamental group of:

Free rank 3 is

π_1(3-leaved rose, $\emptyset$)

Trees

backtrack

Theorem

Let Γ be a labelled, connected, directed graph. The following are equivalent:

1. *Γ has no reduced loops.*
2. *For any vertices $v, w \in \Gamma$, there exists a unique path that starts at v and ends at w.*

Sketch:

Assume Gamma has no reduced loops.

Definition

If Γ satisfies those conditions, we call Γ a **tree**.

So we can shorten p1, p2.
Pf follows by induction.

The Retract Lemma

Theorem

Let Γ be a basepointed X-labelled graph, let T be a tree contained in Γ, and let Γ' be the basepointed labelled graph obtained by replacing T with a single vertex and giving each edge a different label in some alphabet Y. Then $\pi_1(\Gamma, v_0) = \pi_1(\Gamma', v_0)$.

Proof of the Retract Lemma

Loops map to loops, so enough to show that reduced loops map to reduced loops.

Fundamental groups of graphs are free

Corollary

Let Γ be a finite basepointed X-labelled graph with basepoint v_0, E edges, and V vertices. Then $\pi_1(\Gamma, v_0) = F_n$, where $n = E - V + 1$.

Proof:

Same proof works for infinite graphs, but you have to prove things about spanning trees of infinite graphs.

Covering spaces

Let $X = \{a_1, \ldots, a_n\}$.

Definition

We call a completely X-labelled graph a **covering space** of the n-leaved rose:

Coset diagrams

Definition

For $H \leq F_n$, we define the (right) **coset diagram** of H to be the basepointed X-labelled graph given by:

Main Theorem of covering spaces

Let $X = \{a_1, \ldots, a_n\}$.

Theorem

There exists a bijection between subgroups of F_n and basepointed covering spaces of the n-leaved rose, given by:

- **Subgroup to covering space:** *Map H to its coset diagram.*
- **Covering space to subgroup:** *Map (Γ, v_0) to $\pi_1(\Gamma, v_0)$.*

Proof:

Proof of the Kurosh Subgroup Theorem