Background for Math 221B, Spring 2019
(Old Math 128A final)

1. (13 points) Define what it means for $(G, +)$ to be an **additive abelian** group. In other words, define what it means for G to be an abelian group under the operation of $+$.

2. (17 points)

(a) Define what it means for R to be a ring.

(b) Now let R be a ring. Define what it means for a subset S of R to be a sub**ring** of R.

(c) Again let R be a ring. Define what it means for a subset A of R to be an ideal of R.

3. (12 points) Let

$$\alpha = (1\ 7)(2\ 5\ 4\ 8)(3\ 11\ 9\ 6\ 12)$$
$$\beta = (1\ 10\ 7)(2\ 11\ 6\ 3\ 9\ 4)(5\ 12\ 8)$$

Calculate $\alpha\beta$, α^{-1}, and $|\alpha|$. Put all permutation answers in cycle form, and show all your work.

4. (12 points)

(a) List all abelian groups of order $32 = 2^5$, up to isomorphism. Show all your work.

(b) Which abelian groups of order 32 do **not** contain any elements of order 8? Briefly **justify** your answer.

For questions 5–10, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

5. (13 points) **(TRUE/FALSE)** If G is an additive abelian group, H is a subgroup of G, $a \in G$, and $(a+H)$ has order 2 in G/H, then it must be the case that $a+a=0$ in G (where 0 is the identity in G).

6. (13 points) **(TRUE/FALSE)** It is possible there exists a group G of order 520 that has a subgroup isomorphic to A_4 (the alternating group of rank 4).

7. (13 points) **(TRUE/FALSE)** If G is a cyclic group of order 48, then it is possible that G has a subgroup isomorphic to $\mathbf{Z}_4 \oplus \mathbf{Z}_4$.

8. (13 points) **(TRUE/FALSE)** If R is a ring, and S is a subgroup of R under the addition in R, then it must be the case that S is a sub**ring** of R.

9. (13 points) **(TRUE/FALSE)** If $\varphi : G \to \mathbf{Z}_3 \oplus \mathbf{Z}_3$ is a **surjective** (onto) homomorphism, then it must be the case that there exists a normal subgroup N of G such that G/N is a nontrivial abelian group.

10. (13 points) **(TRUE/FALSE)** If H is a normal subgroup of the group G, $a, b \in G$, and $(aH)(bH)^{-1} \neq H$, then it is possible that $a = b$.

11. (17 points) **PROOF QUESTION.** Let G be a group of order $63 = 3^2 \cdot 7$. Prove that G contains an element of order 3.

12. (17 points) **PROOF QUESTION.** Let n be an integer ≥ 3, and let

$$H = \{\alpha \in S_n \,|\, (\alpha(1) = 1 \text{ and } \alpha(2) = 2) \text{ or } (\alpha(1) = 2 \text{ and } \alpha(2) = 1)\}.$$

Prove that H is a subgroup of S_n.

13. (17 points) **PROOF QUESTION.**

(a) Define what it means for a function $\varphi : G \to \overline{G}$ to be onto (i.e., to be surjective).

(b) Define what it means for a function $\varphi : G \to \overline{G}$ to be a homomorphism.

(c) Now let G and $\overline{G}$ be groups, and let $\varphi : G \to \overline{G}$ be an onto (surjective) homomorphism. Prove that if $a \in G$ satisfies $ax = xa$ for all $x \in G$, then $\varphi(a)$ satisfies $\varphi(a)\overline{x} = \overline{x}\varphi(a)$ for all $\overline{x} \in \overline{G}$.

14. (17 points) **PROOF QUESTION.**

(a) Give an example of a ring R in which there exist at least 4 distinct elements $x \in R$ such that $x^3 = x$. (Suggestion: Try $\mathbf{Z}_n$ for a suitable n.)

(b) Prove that if R is an integral domain, then there exist at most 3 distinct elements $x \in R$ such that $x^3 = x$.