Math 221A, problem set 08
Due: Mon Nov 08
Last revision due: Mon Dec 06

Problems to be turned in: Problem x.y.z of Artin denotes problem y.z in Chapter x.

1. This problem establishes some facts that are useful when analyzing Sylow subgroups of a finite group.

   Let G be a finite group, and let H and K be subgroups of G such that $|H| = |K| = p$, where p is prime.

   (a) Prove that either $H \cap K = 1$ or $H = K$.

   (b) Now suppose that $H = \langle a \rangle$, $H \lhd G$, g has order m, and $gag^{-1} = a^k$. Prove that $k^m = 1 \pmod{p}$.

2. Prove that if G is a group of order 153, then G is abelian.

3. Prove that if G is a group of order 56, then G is not simple.

4. Let G be a group of order 105.

   (a) Prove that G either has a normal Sylow 5-subgroup or a normal Sylow 7-subgroup.

   (b) By considering both cases, prove that any Sylow 5-subgroup of G and any Sylow 7-subgroup of G commute with each other, and therefore, that G has a subgroup of order 35. (Suggestion: use problem 1.)

5. Prove that if $H \leq S_n$ and H is not contained in A_n, then H has a normal subgroup of index 2.

6. Let G be a finite group of order n. Recall that the *regular representation* of G is the (transitive) permutation representation $G \to \mathrm{Perm}(G) \approx S_n$ induced by the action of G by left multiplication on its elements. I.e., in the notation of Section 7.1, the regular representation is $g \mapsto m_g$, where $m_g(s) = gs$.

   (a) Prove that for $g \in G$, m_g is a permutation with no fixed points.

   (b) Describe the cycle-shape of m_g when g has order 2.

   (c) Prove that if G has order $2k$ for $k > 1$ odd, then G is not simple.

7. Let G be a group of order $p^k m$, where p is prime and p does not divide m. Suppose also that either

   - $m < p + 1$; or
   - $p + 1$ does not divide m and $m < 2p + 1$.

   Prove that G is not simple.