How to factor polynomials and find their zeros
Math 19

Suppose we have a polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with integer coefficients (the most common situation in a math class), and we either want to find the zeros of $P(x)$ (rational, real, or otherwise) or factor $P(x)$. While in reality, this is generally impossible, here are some guidelines that will work a fair amount of the time in a math class.

Finding zeros is factoring. The first point is that much of the time, the most efficient way to find the zeros of $P(x)$ is to factor it, since if

$$P(x) = a_n(x - c_1)(x - c_2)\dots(x - c_n),$$

then the zeros of $P(x)$ are $c_1, \dots, c_n$. This is not always possible, but when it works, it works quite well.

Degree 2 is done. If $P(x)$ has degree 2, or if you can pull linear factors $(x - c)$ out of $P(x)$ until what's left has degree 2, then you're done, as any polynomial of degree 2 can be factored using the quadratic formula.

List of guesses. In a math class, it is often (but not always) the case that all or most of the zeros of $P(x)$ will be rational. Therefore, one good starting place is to make a list of the possible rational zeros of $P(x)$, which, by the Rational Zeros Theorem, is the set of all numbers of the form $\pm\frac{p}{q}$, where p divides a_0 (the constant coefficient) and q divides a_n (the leading coefficient).

A step-by-step repeated procedure. With the list of possible rational zeros in hand, we can apply the following step-by-step procedure.

1. For each possible rational zero c, calculate $P(c)$.
2. If $P(c) \neq 0$, go back to step 1 for the next c in the list of guesses, until you run out of guesses.
3. If $P(c) = 0$, then c is a zero of $P(x)$, and $(x - c)$ divides $P(x)$. In that case, calculate $Q(x) = \dfrac{P(x)}{x - c}$, and start over with factoring $Q(x)$, possibly with a new list of guesses. (You can eliminate any values of c that have previously been eliminated as zeros, but be careful; c may be multiple zero of $P(x)$, so it may still be a zero of the new quotient $Q(x)$.)

If you like synthetic division, note that you can calculate $Q(x) = \dfrac{P(x)}{x - c}$ and $P(c)$ in one step, as $P(c)$ is just the remainder that you get when you divide $P(x)$ by $(x - c)$. For examples in the text that follow the above procedure, see pp. 273–275.

Now, if you're not experienced with factoring, the "start over" part of step 3 may look like it will take a long time. Here's why that's not really the case, especially in a math class.

- If the degree of $P(x)$ isn't too high, because you're basically done when you reduce to degree 2, it may actually only take a few steps to finish. In comparison, especially because of the $\pm$, you might have to try lots of cases (see below).

- In a math class, it is likely that the teacher will try to make questions reasonable, and choose small numbers like ± 1 to be the zeros.
- Just trying all of the rational zeros, without dividing, will never find any non-rational zeros.

Finding rational zeros without factoring. If you only want to find all *rational* zeros, then one way to do that is to calculate $P(c)$ for all values of c on the $\pm\frac{p}{q}$ list. You can stop once either:

1. You find n different zeros, as $P(x)$ can have at most n zeros; or
2. You go through all values of $c = \pm\frac{p}{q}$.

The problem with this is that, for example, if $P(x)$ has any zeros of multiplicity greater than 1, you won't actually ever find n different zeros, which means that you'll end up in case 2. And that can mean trying a lot of cases: For example, if you're looking for all rational zeros of a polynomial of the form $x^5 + \cdots + 24$, there are 16 cases $(\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24)$ to check; if the polynomial has the form $7x^5 + \cdots + 24$, there are 32 cases; and so on. In general, especially working by hand in a timed situation, factoring will probably work better.