How to solve recurrence relations
(especially with generating functions)
Math 142
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This handout summarizes our methods for solving the following problem under certain specific circumstances.

> *Given a recurrence relation for a_n and a sufficient number of initial conditions, find a closed formula for a_n.*

Linear homogeneous recurrences. Given a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_{k-1} a_{n-(k-1)} + c_k a_{n-k},$$

we form the associated *characteristic equation*

$$x^k = c_1 x^{k-1} + c_2 x^{k-2} + \cdots + c_{k-1} x + c_k$$

and find solutions $\alpha_1, \ldots, \alpha_k$, which may not all be real numbers, and may have repeated solutions.

In the most straightforward case where the solutions $\alpha_1, \ldots, \alpha_k$ are all different/distinct, we can use generating functions to prove that

$$a_n = b_1 \alpha_1^n + \cdots + b_k \alpha_k^n,$$

where the coefficients b_i can be solved for using any k initial conditions, e.g., $a_0, \ldots, a_{k-1}$. (If there are repeated roots, something similar but more complicated works.)

Linear inhomogeneous recurrences. For a recurrence of the form $a_n = ca_{n-1} + f(n)$, where $c \neq 1$, we have solutions for the following special cases of $f(n)$:

Recurrence ($c \neq 1$)	Solution form
$a_n = ca_{n-1} + d$	$a_n = Ac^n + B_0$
$a_n = ca_{n-1} + dn + e$	$a_n = Ac^n + B_1 n + B_0$
$a_n = ca_{n-1} + dn^2 + en + f$	$a_n = Ac^n + B_2 n^2 + B_1 n + B_0$

Indeed, the analogous result holds for any polynomial function $f(n)$, as long as $c \neq 1$.

The method of generating functions. Both of the above methods can be dervied (i.e., proven as theorems) using generating functions. To simplify the general situation a bit, assume we have a recurrence relation of the form

$$a_n = ca_{n-1} + f(n),$$

where this time, we allow the possibility that $c = 1$. We then apply the following general method to solve for $g(x) = \sum_{n=0}^{\infty} a_n x^n$, the generating function of the sequence a_n.

1. *Multiply both sides by x^n.* We get:

$$a_n x^n = ca_{n-1} x^n + f(n) x^n.$$

2. *Sum both sides from $n = 1$ to ∞.* We get:

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} c a_{n-1} x^n + \sum_{n=1}^{\infty} f(n) x^n.$$

3. *Adjust the numbering to express all a_* sums in terms of $g(x)$.* This is the fussiest/most delicate part of the procedure, as you have to use slightly different tactics for each term.

    - On the left-hand side, the sum is numbered correctly but is missing the a_0 term. Therefore, the left-hand side can be rewritten

    $$\sum_{n=1}^{\infty} a_n x^n = \left(\sum_{n=0}^{\infty} a_n x^n \right) - a_0 = g(x) - a_0.$$

    - For the first term on the right-hand side, the sum actually has all of the correct terms in it (as we'll see) but is numbered incorrectly. To fix the numbering, we make the substitution $k = n - 1$, or $n = k + 1$, to get

    $$\sum_{n=1}^{\infty} c a_{n-1} x^n = \sum_{k=0}^{\infty} c a_k x^{k+1} = cx \sum_{k=0}^{\infty} a_k x^k = cxg(x).$$

    Moving the a_0 term to the right-hand saide and combining it with the sum $\sum_{n=1}^{\infty} f(n) x^n$, we get an equation

    $$g(x) = cxg(x) + h(x),$$

    where $h(x) = a_0 + \sum_{n=1}^{\infty} f(n) x^n$ encodes both the initial conditions and the inhomogeneous term $f(x)$. Solving for $g(x)$ (algebra omitted), we get

    $$g(x) = \frac{h(x)}{1 - cx}.$$

Once we have the generating function $g(x)$, we can sometimes find a closed formula for a_n. Some of the standard generating function formulas may be useful:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{1}$$

$$\frac{1}{(1-x)^m} = \sum_{n=0}^{\infty} \binom{n + (m-1)}{m-1} x^n \tag{2}$$

Also, starting with (**??**) and applying the operator $x \frac{d}{dx}$ (i.e., differentiate and then multiply by

x) repeatedly, we get

$$x(1-x)^{-2} = \sum_{n=1}^{\infty} nx^n = \sum_{n=0}^{\infty} nx^n \tag{3}$$

$$x(1-x)^{-2} + 2x^2(1-x)^{-3} = \sum_{n=1}^{\infty} n^2x^n = \sum_{n=0}^{\infty} n^2x^n \tag{4}$$

$$x(1-x)^{-2} + 6x^2(1-x)^{-3} + 6x^3(1-x)^{-4} = \sum_{n=1}^{\infty} n^3x^n = \sum_{n=0}^{\infty} n^3x^n \tag{5}$$

and so on.

Coefficient shifting. One other useful technique for dealing with generating functions is accounting for what happens when you take a known generating function and multiply by x^k.

To start with a concrete example, we know that

$$\frac{1}{(1-x)^7} = \sum_{n=0}^{\infty} \binom{n+6}{6} x^n.$$

If we obtain a generating function

$$g(x) = \frac{x^4}{(1-x)^7} = \sum_{n=0}^{\infty} \binom{n+6}{6} x^{n+4} = \sum_{k=0}^{\infty} a_k x^k,$$

how can we find a formula for a_k?

Answer: Make the substitution $k = n + 4$, which gives us

$$\sum_{k=4}^{\infty} \binom{k+2}{6} x^k = \sum_{k=0}^{\infty} a_k x^k.$$

We see that $a_0 = a_1 = a_2 = a_3 = 0$, since there are no terms lower than x^4 on the left-hand side; and for $k \geq 4$, $a_k = \binom{k+2}{6}$.

The general result is that multiplying $\sum_{n=0}^{\infty} a_n x^n$ by x^m has two effects:

1. The coefficients are shifted m steps to the right, i.e., the nonzero coefficients can only start with the x^m term; and

2. If the old coefficients are a_n and the new coefficients are b_k, then for $k \geq m$, $b_k = a_{k-m}$.

However, it's easy to make a sign/directional error if you apply that general result, so at least for beginners, it's probably more reliable just to do a substitution similar to the $k = n+4$ substitution we did in our example.