Math 142, problem set 08
REVISED/CORRECTED WED NOV 02
Outline due: Thu Nov 03
Final version due: Tue Nov 08

Problems to be turned in:

1. For this problem, you need to know a fact from what you might call "K–12 number theory", if not expressed in K–12 terms: Namely, that every positive integer n has a unique factorization of the form

$$n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k},$$

where $1 < p_1 < p_2 < \cdots < p_k$ are prime. Recall also that $a^{-s}b^{-s} = (ab)^{-s}$.

   (a) If you write $\dfrac{1}{1-2^{-s}}$ as a sum of the form $\sum n^{-s}$, which n appear as the base of some n^{-s} in the sum? Explain.

   (b) If you write $\left(\dfrac{1}{1-2^{-s}}\right)\left(\dfrac{1}{1-3^{-s}}\right)$ as the product of two sums of the form described above, and multiply those sums together to get a single sum $\sum n^{-s}$, which n appear as the base of some n^{-s} in the final sum? Explain in terms of the prime factorizations of those n.

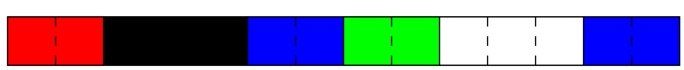
   (c) Explain why (prove that)

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$

2. Find a recurrence relation for the number of sequences of pennies, nickels, dimes, and quarters that add up to n cents.

3. Find a recurrence relation for the number of sequences of letters A–Z of length n that do not contain the sequence EXIT.

4. Find a recurrence relation for the number of tilings of an $n \times 1$ grid using red, green, and blue dominoes (2×1 tiles) and black and white triominoes (3×1 tiles). For example, one such tiling for $n = 14$ is:



5. Find a recurrence relation for the number of sequences of the numbers 1, 2, and 4 (possibly with repeats) such that the sum of the numbers in the sequence is n and the sequence does not contain "1, 2, 4" as a subsequence.

(continued on next page)

6. Let $a_{n,k}$ be the number of ways to express $\{1, \ldots, n\}$ as the (unordered) union of k nonempty pairwise disjoint sets. For example, $a_{3,2} = 3$ because there are exactly 3 ways to express $\{1, 2, 3\}$ as the union of two nonempty disjoint sets:

$$\{1,2,3\} = \{1\} \cup \{2,3\} = \{1,3\} \cup \{2\} = \{1,2\} \cup \{3\}.$$

Find a recurrence relation for $a_{n,k}$. (Suggestion: Do a case breakdown based on whether the last element n is contained in a singleton set.)

7. The merge sort algorithm `merge` does the following when applied to a list of n numbers:

- Apply `merge` to the first $n/2$ entries in the list to produce a sorted list of length $n/2$.
- Apply `merge` to the last $n/2$ entries in the list to produce a sorted list of length $n/2$.
- Merge the two sorted lists to produce a sorted list of length n.

Suppose that last step takes n units of time. Let a_n be the amount of time it takes to apply `merge` to a list of n numbers, taking $a_1 = t$ for some constant t.

(a) Find a recurrence relation for a_n, assuming n is a power of 2.

(b) Solve for a_n, using the table in Section 7.2.