Math 142, problem set 06
Outline due: Wed Oct 12
Final version due: Mon Oct 17

For all of the following problems, explain/justify your answer, and write your final numerical answer as a sum or product of factorials, $\binom{n}{k}$, and so on.

Problems to be turned in:

1. The Roman (English) alphabet has 26 letters and the (modern) Greek alphabet has 24 letters.
   (a) How many ways are there to make a 7-letter word from Roman letters, a 5-letter word from Greek letters, and a 6-letter word from some mixture of Roman and Greek letters, where no letter appears twice (even in two different words)? Note that when we say "word", the order of letters in the word matters, e.g., ABCDE is different from CBDEA.
   (b) How many ways are there to choose a 7-letter subset of the Roman alphabet, a 5-letter subset of the Greek alphabet, and a 6-letter subset of the remaining unused letters in the Roman and Greek alphabets (i.e., no letter appears in two of the subsets)? Note that as usual, order does not matter within a set, e.g., $\{A, B, C, D, E\} = \{C, B, D, E, A\}$.

2. Model the following situations as balls in boxes, specifying any constraints on repetition, and specifying whether the balls are distinct or identical.
   (a) Arrangements of the word INITIATION.
   (b) Nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 = 75$, with $x_2 \geq 5$ and $15 \leq x_3 \leq 31$.
   (c) Selections of pennies, nickles, dimes, and quarters, with an even number of pennies and an odd number of quarters.

3. (5.5) 32.

4. (a) (5.5) 14(c).
   (b) (5.5) 14(e).

5. Prove that for any integer $n \geq 4$, we have:

$$\binom{n}{1}+n\binom{4}{3}(n-1)+\binom{4}{2}\left(\frac{n(n-1)}{2}\right)+n\binom{4}{2}\binom{2}{1}\left(\frac{(n-1)(n-2)}{2}\right)+\frac{n!}{(n-4)!}=n^4.$$

   For full credit, don't just expand the polynomials; find an explanation that shows where this identity comes from and suggests how it might be generalized.

(continued on next page)

6. Build a generating function for a_r, the number of integer solutions to the equations:

   (a) $e_1 + e_2 + \cdots + e_8 = r$, $1 \leq e_i \leq 7$.

   (b) $e_1 + e_2 + e_3 + e_4 = r$, $2 \leq e_1$, $0 \leq e_2, e_3 \leq 5$, $e_4 \geq 0$, e_4 is a multiple of 3.

7. Build a generating function for a_r, the number of distributions of r identical objects into:

   (a) Four different boxes with between five and twelve objects in each box.

   (b) Five different boxes with up to three objects in the first box and at least four objects in each of the other boxes.

   (c) Six different boxes with an odd number of objects in the first box, an even number of objects in the second box, and no restriction on the number of objects in the other boxes.