Math 142, problem set 05
First draft: Mon Oct 07
Final version: Fri Oct 11

Problems to be done, but not turned in: (3.2) 3, 5, 11, 13; (3.3) 5, 9; (3.4) 1.

Problems to be turned in:

1. (3.2) 12.
2. (3.2) 16(a).
3. (3.3) 6. (Make sure you explain the difference(s) between assignment and TSP.)
4. (3.4) 4. (Answers will vary.)
5. In this problem, we fix a connected graph G, and we think of G as a *resistor network*, with edges representing resistors, and vertices representing terminals (sites where resistors are connected). To simplify matters, we ignore batteries, currents, and resistances, and concentrate on voltages. We also direct the edges of G, though we ignore the directions for the purposes of paths, circuits, etc; we have the directions solely for the purpose of defining the signs in the formulas below.

   We define a *flow* $f(e)$ on G to be a real-valued function on the edges of G (i.e., $f(e)$ is a real number for any edge e). If $f(e)$ is a flow on G, then we say that a real-valued function $V(x)$ on the vertices of G is a *potential function* for $f(e)$ if, for any directed edge e going from x_0 to x_1, $V(x_0) - V(x_1) = f(e)$. In other words, $V(x)$ is a potential function for $f(e)$ exactly if $f(e)$ describes the *voltage drop* that occurs when travelling over e in the indicated direction. (The electrically inclined may recognize that by Ohm's Law, $f(e)$ is essentially the current over e, adjusted for resistance.)

   (a) Show that if a flow $f(e)$ has a potential function $V(x)$, then the *voltage drop* (defined below) around any circuit, adjusting for edge directions, is 0. More precisely, if we travel around a circuit in G, and denote "forward" edges by e_i and "backward" edges by e'_j, then

   $$\text{voltage drop} = \sum_i f(e_i) - \sum_j f(e'_j) = 0.$$

   (b) Assuming G is a tree, describe a method for finding a potential function for *any* flow on G. (Corollary: any flow on a tree has a potential function.)

   (c) We now drop the assumption that G is a tree, but assume that $f(e)$ is a flow on G such that the voltage drop around any circuit (as defined in part 5a) is 0. Show that if T is a spanning tree of G, and you apply part 5b to obtain a potential function for $f(e)$ on T, then you actually get a potential function for $f(e)$ on all of G. (Corollary: A flow on a graph has a potential function if and only if the total flow around any circuit is 0. Starting point: How could the potential function on T not be a potential function on G?)