Paths, circuits, trails, cycles
Math 142

In this handout, we go through the distinctions between paths, circuits, trails, and cycles carefully, once and for all. (Inadvertently, we probably also show why being truly careful about these matters can be distracting.)

Conventions. Throughout, G is an undirected graph. We write the (undirected) edge between vertices $a, b \in G$ as (a, b), with the convention that $(a, b) = (b, a)$.

Definition 1. A *path* is a sequence $x_0, x_1, \ldots, x_n$ of vertices of G such that:

1. For $0 \leq i \leq n-1$, x_i is adjacent to x_{i+1} (i.e., (x_i, x_{i+1}) is an edge of G), and
2. For $i < j$, $x_i \neq x_j$, with the possible exception that we may have $x_0 = x_n$, if $n > 2$. (In other words: No repeated vertices, except that we are allowed to end where we started.)

The *length* of the path $x_0, x_1, \ldots, x_n$ is n; that is, the length of a path is the number of edges in the path. The path $x_0, x_1, \ldots, x_n$ is called a *circuit* if $n > 0$ and $x_0 = x_n$ (which can only happen when $n > 2$).

Definition 2. A *trail* is a sequence $x_0, x_1, \ldots, x_n$ of vertices of G such that:

1. For $0 \leq i \leq n-1$, x_i is adjacent to x_{i+1} (i.e., (x_i, x_{i+1}) is an edge of G), and
2. For $i < j \leq n-1$, $(x_i, x_{i+1}) \neq (x_j, x_{j+1})$. (In other words: No repeated edges, even ones travelled in the opposite direction.)

As with paths, the *length* of the trail $x_0, x_1, \ldots, x_n$ is n (the number of edges in the trail), and the trail $x_0, x_1, \ldots, x_n$ is called a *cycle* if $n > 0$ and $x_0 = x_n$.

Remark 3. As an exercise in the trickiness of doing these definitions correctly, the reader may wish to verify that the sequence x_0 is both a path and a trail of length 0 (!). Note also that if we did not explicitly include the condition $n > 0$ in the definitions of circuits and cycles, then x_0 would also be both a circuit and a cycle of length 0 (!!).

Remark 4. Note that Definition 1 carries over verbatim to directed graphs, and has essentially the same meaning in that context. Definition 2 actually also carries over to directed graphs, but has a slightly different meaning: The condition $(x_i, x_{i+1}) \neq (x_j, x_{j+1})$ allows a trail to go between a pair of vertices once in each direction, if corresponding edges exist.

We show that these definitions are "correct" (finally!) by carefully proving a number of "obvious" facts about paths, circuits, trails, and cycles. We begin with a relatively easy fact, about which we still need to be relatively careful.

Theorem 5. *Every path is a trail, and every circuit is a cycle.*

Proof. Since the extra conditions $n > 0$ and $x_0 = x_n$ are identical for circuits and cycles, it is enough to show that every path is a trail. So let $x_0, x_1, \ldots, x_n$ be a path. We first observe that for $i < j \leq n-1$, $(x_i, x_{i+1}) = (x_j, x_{j+1})$ if and only if either $x_i = x_j$ and $x_{i+1} = x_{j+1}$, or $x_i = x_{j+1}$ and $x_{i+1} = x_j$. On the one hand, having no repeated vertices except possibly $x_0 = x_n$ implies that for $i < j \leq n-1$, $x_i \neq x_j$. Furthermore, while it is possible that $x_i = x_{j+1}$, that can only happen if $i = 0$, $j + 1 = n$, and $n > 2$, in which case $x_{i+1} = x_1 \neq x_{n-1} = x_j$. Therefore, the path $x_0, x_1, \ldots, x_n$ satisfies the no repeated edges condition of Definition 2, which means that $x_0, x_1, \ldots, x_n$ is a trail. $\square$

Another relatively straightforward fact is:

Theorem 6. *Every cycle (and therefore, every circuit) has length at least 3.*

Proof. By definition, a cycle cannot have length 0. A trail x_0, x_1 cannot be a cycle because a graph has no self-edges. Finally, if x_0, x_1, x_2 is a trail, then $(x_0, x_1) \neq (x_1, x_2)$, which means that $x_0 \neq x_2$. Therefore, a trail of length 2 cannot be a cycle. □

The next fact is somewhat trickier.

Theorem 7. *Every cycle contains a consecutive subsequence that forms a circuit.*

In other words, every cycle contains at least one circuit "within" itself.

Proof. First, we observe that it is enough to show the following claim:

> **Claim:** Every cycle of length n contains a subsquence $(a, \ldots, a)$ with no repeated vertices besides a.

We proceed by induction on $n \geq 3$. In the base case $n = 3$, any cycle of length 3 must have the form (a, b, c, a). Applying the no repeated edges condition, we see that $(a, b) \neq (b, c)$, $(a, b) \neq (c, a)$, and $(b, c) \neq (c, a)$, which implies that $a \neq c$, $b \neq c$, and $b \neq a$, respectively. Therefore, there are no repeated vertices, and (a, b, c, a) is a circuit.

In the general case, suppose that the Claim is true for all cycles of length strictly less than n, and let $(x_0, x_1, \ldots, x_n)$, $x_0 = x_n$, be a cycle of length n. We then have two cases:

1. If there are no repeated vertices except $x_0 = x_n$, then $(x_0, x_1, \ldots, x_n)$ is itself a circuit.

2. If $x_i = x_j$ for some $0 \leq i < j \leq n$, $(i, j) \neq (0, n)$, then $(x_i, \ldots, x_j)$ is a cycle of length strictly smaller than n, since the two conditions of Definition 2 are met *a fortiori*. Therefore, by induction, this smaller cycle $(x_i, \ldots, x_j)$ containes the desired subsequence, and the theorem follows. □

Exercises: If you are interested in further aspects of getting these definitions exactly right, try the following exercises.

1. Show that if there is a path from a to b, then there is also a trail from a to b.

2. Show that if there is a trail from a to b, then there is also a path from a to b.

3. Show that if there is a path from a to b and a path from b to c, then there is a path from a to c.

4. Show that if there is a trail from a to b and a trail from b to c, then there is a *path* from a to c.

5. Which of exercises 1–4 also hold for directed graphs? Prove the ones that do.