Coefficients of generating functions

$$\frac{1-x^m}{1-x} = 1 + x + x^2 + \cdots + x^{m-1} \tag{1}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \tag{2}$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n \tag{3}$$

$$(1-x^m)^n = \binom{n}{0} - \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \cdots + (-1)^n\binom{n}{n}x^{nm} \tag{4}$$

$$\frac{1}{(1-x)^n} = 1 + \binom{(n-1)+1}{1}x + \binom{(n-1)+2}{2}x^2 + \ldots \tag{5}$$

Note that taking powers of (1) gives a product of the form (4) and the form (5). We can also get new generating functions by substituting for x; for example, replacing x with x^3 in (2) gives

$$\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + \ldots$$

Products: In the abstract, if

$$f(x) = \sum a_k x^k$$
$$g(x) = \sum b_\ell x^\ell,$$
$$h(x) = f(x)g(x) = \sum c_r x^r,$$

then

$$c_r = a_0 b_r + a_1 b_{r-1} + \cdots + a_{r-1} b_1 + a_r b_0 = \sum_{k=0}^{r} a_k b_{r-k}. \tag{6}$$

In practice, we are usually less interested in the abstract summation form (6), and more likely to use the following idea: In the product $f(x)g(x)$, where

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$
$$g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots$$

to find the coefficient of x^r, we sum all products $(a_k x^k)(b_\ell x^\ell)$ where $k + \ell = r$.

This approach works particularly well if many a_k and b_ℓ are 0. For example, suppose

$$f(x) = 7x^3 + 5x^4 + 4x^{10} + 3x^{13} + \ldots$$
$$g(x) = 2 + 4x^7 + 6x^9 + 7x^{12} + \ldots$$

and we want to find the coefficient of x^{13} in $f(x)g(x)$. By starting with smaller k and larger ℓ and increasing k and decreasing ℓ to match, we see that the only pairs (k,ℓ) such that both a_k and b_ℓ are nonzero and $k + \ell = 13$ are the pairs $(k,\ell) = (4,9)$ and $(k,\ell) = (13,0)$. It follows that the coefficient of x^{13} in $f(x)g(x)$ is $5(6) + 3(2) = 36$.