Binomial identities
Math 142

Part A: Answer all of the following questions. Try to think in terms of a process: What kind of process would have this number of possible outcomes?

1. For $k \le 6 \le n$, what might $\binom{n}{6}\binom{6}{k}$ be counting?
2. For $k \le n$, what might $\binom{n}{k}2^k$ be counting?
3. What might n^4 be counting? (Think balls in boxes.)
4. For $3 \le m \le n$, what might $\binom{n}{3}\binom{n}{m-3}$ be counting?
5. For $3 \le m \le n$, what might $\binom{n}{3}\binom{n-3}{m-3}$ be counting? Find a different product of binomials equal to $\binom{n}{3}\binom{n-3}{m-3}$.

Part B: Do one of the following problems in your team. In each problem, use a combinatorial proof as suggested by the ideas from Part A.

1. Prove that
$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+r}{r} = \binom{n+r+1}{r}.$$
2. Prove that
$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$
3. Prove that
$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.$$
4. Prove that
$$\sum_{k=0}^{m} \binom{m}{k}\binom{n}{r+k} = \binom{m+n}{m+r}.$$