In-class groupwork activities

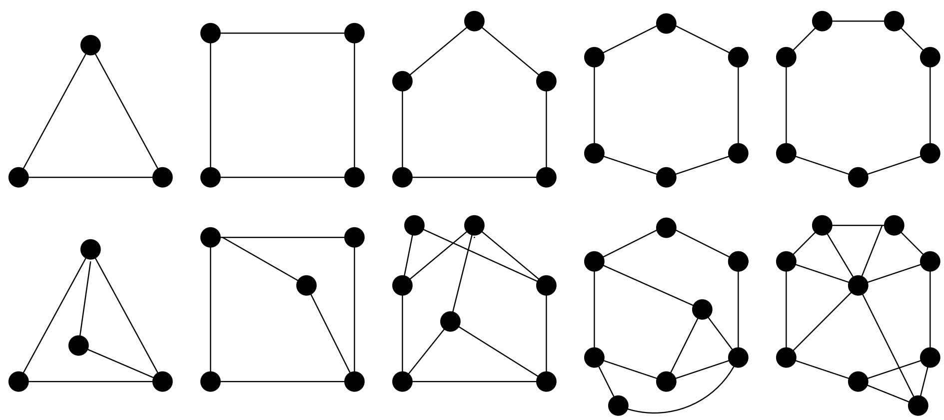
Introduction to combinatorics

Teamwork roles:

- The **facilitator** organizes the team to make sure the task is complete and makes sure all team members stay on task.
- The **documenter** writes the team's work on the board.
- The **presenter** talks through the team's work to the entire class at the end.
- (if a team of four) The **verifier** makes sure that everyone on the team understands the team's final answer.

1.1: We say that a graph G can be **2-colored** if the vertices of G can be colored either black or white such that no two adjacent vertices have the same color.

1. Which of the following graphs can be 2-colored? Explain.



2. Based on your work in part 1, find a **necessary** condition on a graph for it to be 2-colorable. Explain.
3. Find a method (an algorithm) for 2-coloring a graph. Keep two questions in mind as you do this:
   - How can you be sure your method works if the condition in 2 holds for a particular graph?
   - What happens if the condition you found in part 2 doesn't hold for a particular graph?

1.2: In your groups/teams:

1. Make up four connected graphs, each with the same number of vertices (6,7, or 8), such that two of your graphs are isomorphic (in some non-obvious way), and there are no other isomorphisms among those four graphs.
2. Now switch as directed, look at some other team's graphs, and figure out which two are isomorphic. You should be able to explain how you know the two isomorphic graphs are isomorphic, and why the other two graphs are not isomorphic either to the isomorphic pair or each other.

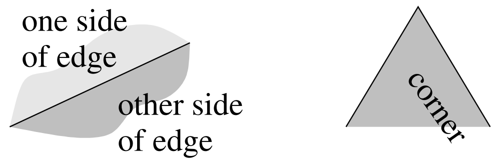
1.3/1.4:

1. Draw some graphs where every vertex has degree 2. What do you notice about the number of edges and vertices in each graph?
2. Draw some graphs where every vertex has degree 3. What do you notice?
3. Draw some graphs where every vertex has degree 4, degree 5, etc. What do you notice? Generalize.
4. **Planar** graphs can be drawn on the board without having edges cross. Can you find planar graphs where every vertex has degree 2? Non-planar graphs where every vertex has degree 2? Same for degrees 3, 4, 5, ...

1.4: Consider the following conditions on a (hypothetical) planar graph. Write down the conditions assigned to your team, and try to invent a planar graph satisfying those conditions.

1. Every region is a triangle (3-sided), and regions meet at most 3 to a vertex.
2. Triangles, at most 4 meeting at a vertex.
3. Triangles, at most 5 meeting at a vertex.
4. Triangles, at **least** 6 meeting at a vertex.
5. Squares, at most 3 meeting at a vertex.
6. Pentagons, at most 3 meeting at a vertex.
7. Hexagons, at most 3 meeting at a vertex.
8. 7-gons, at **least** 3 meeting at a vertex.

Now suppose you have a hypothetical planar graph G satisfying your team's conditions, with $\mathbf{r}$ regions, $\mathbf{e}$ edges, and $\mathbf{v}$ vertices.



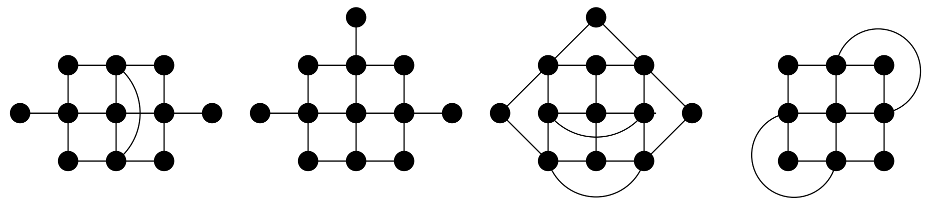
A. Note that because G is planar, you can draw it so that every edge has two sides (left-hand picture above). Count the number of sides of edges in two different ways to get a equation involving $\mathbf{r}$ and $\mathbf{e}$.

B. Next, count corners of regions (right-hand picture above) in two different ways to get an **inequality** involving $\mathbf{r}$ and $\mathbf{v}$.

C. What kind of **inequality** or inequalities can you get by combining the previous two conditions and $\mathbf{r} - \mathbf{e} + \mathbf{v} = 2$?

D. Can you draw a graph that achieves the largest/smallest value of the inequality/inequalities you just found? (Don't forget the "outside" region.)

2.1: Consider the following graphs.



1. For each graph, if possible, find a path that traverses every edge of the graph exactly once, or explain why no such path can exist. (Think: Where must such a path start or end?) Such a path is called an **Euler trail**.

2. For each graph where you found an Euler trail, can you find such a path that begins and ends at the same place? That kind of path is called an **Euler cycle**.

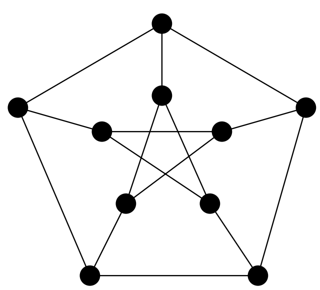
3. Find criteria, in terms of degrees of vertices, that describes exactly when a connected graph has an Euler trail, and when a connected graph has an Euler cycle.

1.3, 1.4, 2.1, 2.2: Let G be a connected graph. For each of the following properties, write down the definition of what it means for G to have that property, and if possible, write down a test or tests for when G has that property, and when G doesn't have that property.

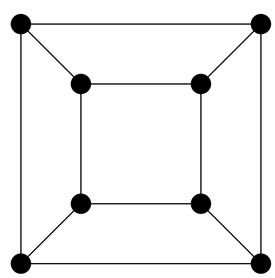
1. G is bipartite.
2. G is planar.
3. G has an Euler cycle.
4. G has an Euler trail.
5. G has a Hamilton circuit.
6. G has a Hamilton path.

Chs. 1–2 review: For each of the following graphs G:

1. Determine if G is bipartite; if it isn't, explain how you know it isn't.
2. Determine if G is planar; if it isn't, find a $K_{3,3}$ or K_5 configuration in it.
3. Determine if G has an Euler cycle, or an Euler trail but no Euler cycle, or no Euler trail. Justify your answer.
4. Determine if G has a Hamilton cycle. Carefully justify your answer if it doesn't.
5. Determine the chromatic number of G. Carefully justify why G cannot be colored using fewer colors.



(a)



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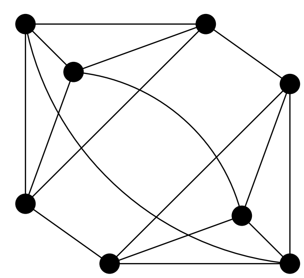

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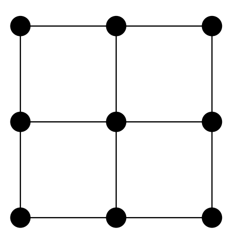

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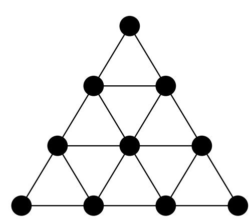

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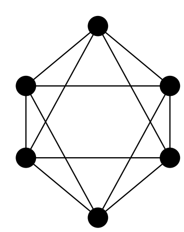

(f)

5.1: Go to the class web page:

`www.timhsu.net/courses/142`

Click on the link "Introduction to counting" to view the handout with today's problems. Do all of the problems in your group, and as you solve them at the board:

- The documenter (heart card) will save the answers on paper as you go along.
- When you present your work, **EXPLAIN** your reasoning for counting the set you're supposed to count. What process produces an arbitrary element/all elements of that set?
- Write your final numerical answer as a product or sum.

5.2: Go to the class web page:

`www.timhsu.net/courses/142`

Click on the link "Arrangements and selections (slots)" to view the handout with today's problems. Do all of the problems in your group, and as you solve them at the board:

- The documenter (heart card) will save the answers on paper as you go along.
- When you present your work, **EXPLAIN** your reasoning for counting the set you're supposed to count. What process produces an arbitrary element/all elements of that set?
- Write your final numerical answer as a product or sum, possibly involving $\binom{n}{k}$.

5.3: Go to:

`www.timhsu.net/courses/142`

Click on the link "Arrangements and selections (dividers)" to view the handout with today's problems. Do all of the problems in your group, and as you solve them at the board:

- The documenter (heart card) will save the answers on paper as you go along.
- When you present your work, **EXPLAIN** your reasoning for counting the set you're supposed to count. What process produces an arbitrary element/all elements of that set?
- Write your final numerical answer as a product or sum, possibly involving $\binom{n}{k}$.

5.4: Go to:

`www.timhsu.net/courses/142`

Click on the link "Distributions" to view the handout with today's problems. Do one problem in your group, as directed, and as you solve it at the board:

- The documenter (heart card) will save the answers on paper as you go along.
- When you present your work, **EXPLAIN** your reasoning for counting the set you're supposed to count. What process produces an arbitrary element/all elements of that set?
- Write your final numerical answer as a product or sum, possibly involving $\binom{n}{k}$.

5.5: Go to:

`www.timhsu.net/courses/142`

Click on the link "Binomial identities" to view the handout with today's problems. Work in your group as directed.

6.1:

1. Expand (multiply out)
$$(x^0 + x^5)(y^1 + y^3 + y^5 + y^7)(z^0 + z^3 + z^6).$$
(Pre-check: How many monomial terms will you get in the end?)

2. Now substitute x for both y and z (i.e., set $y = x$, $z = x$) and simplify (collect terms).

3. Let a_r be the coefficient of x^r in part 2. Can you find an interpretation for a_r in terms of an integer-solution-to-an-equation problem? How about a balls-in-boxes problem? In either language, why is the value of a_6 what it is, combinatorially?

6.1: Find generating function models for the following.

1. The number of ways to choose r children's office dentist office toys if we are allowed to choose among up to 6 plastic spiders, up to 6 sticker kits, between 4 and 10 putty packs and between 2 and 8 squishy mice.
2. The number of ways to choose a total of r soccer players from among 23 teams, each consisting of 11 identical players. (I.e., we do not care about *which* players are chosen from a given team, only how many.)
3. The number of ways to choose a bouquet of r flowers if we have at least two roses, at least three poppies, and an odd number of chrysanthemums.
4. Assuming that we have a villain pool consisting of 7 unique super-villains and large numbers of identical red, green, and blue henchmen, the number of ways to choose r bad guys (super-villains plus henchmen).

6.2: Find the indicated coefficients of the following generating functions.

1. Coefficient of x^{35} in $(1+x^6)(x^2+x^3+x^4+\dots)^5$.
2. Coefficient of x^{17} in $(1+x)^{31}(1+x+x^2+x^3)^5$.
3. Coefficient of x^{37} in $(x+x^2+x^3)(1+x^3+x^6+x^9+\dots)^4$.
4. Coefficient of x^{32} in $(x+x^2+x^3+x^4+x^5)^{13}$.
5. Coefficients of x^{41} and x^{42} in $(1+x^2+x^4+x^6+\dots)^5$.
6. Coefficients of x^{23} and x^{24} in $\dfrac{1}{(1+x)^7}$.

6.2: Use generating functions to enumerate the following.

1. The number of ways to choose 23 children's office dentist office toys if we are allowed to choose among up to 6 plastic spiders, up to 6 sticker kits, between 4 and 10 putty packs and between 2 and 8 squishy mice.
2. The number of ways to choose a total of 16 soccer players from among 23 teams, each consisting of 11 identical players. (I.e., we do not care about *which* players are chosen from a given team, only how many.)
3. The number of ways to choose a bouquet of 24 flowers if we have at least two roses, at least three poppies, and an odd number of chrysanthemums.
4. Assuming that we have a villain pool consisting of 7 unique super-villains and large numbers of identical red, green, and blue henchmen, the number of ways to choose 37 bad guys (super-villains plus henchmen).

6.3:

1. (a) Find the generating function for the number of ways to make r cents from pennies, nickels, dimes, and quarters.
   (b) Compute your generating function up through the x^{24} term.
2. (a) Find the generating function for the number of ways to express r as the sum of two odd primes (3, 5, 7, 11, 13, ...).
   (b) What can you say about a_r when r is odd?
   (c) The *Goldbach conjecture* posits that every even number ≥ 6 is the sum of two odd primes. How would you express that in terms of your generating function?

7.1: Warmup: Write out the first 10 terms of the following sequences defined by recurrence relations.

1. $a_1 = a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$.
2. $a_1 = 1$, $a_2 = 3$, $a_n = a_{n-1} + a_{n-2}$.
3. $a_1 = 1$, $a_n = 2a_{n-1} + 1$.
4. $a_1 = 2$, $a_n = a_{n-1} + n$.

7.1: Find recurrence relations for the following sequences.

1. The number of $2 \times n$ Lego walls of height 1 made from red 2×1 bricks and blue 2×4 bricks.
2. The number of sequences of \$1, \$2, \$5, and \$10 bills whose total is n dollars.
3. The number of sequences of English letters not containing the word HAMLET.
4. The number of ways for one team in American football to score n points, assuming that a team can score 2 points (safety), 3 points (field goal), 6 points (touchdown), 7 points (touchdown with extra point), and 8 points (touchdown with two-point conversion).

7.1–7.2: Find recurrence relations for the following sequences.

1. $a_{n,k}$, the number of ways to put n red, green, blue, and yellow balls into k (distinct) boxes, with between 4 and 6 balls in each box. (Suggestion: How many balls go in the first/last box?)

2. w_n, the number of length n sequences of English letters that contain the word HAMLET at least once.

3. b_n, the number of decimal digit sequences (0–9) of length n with no consecutive 1's and not ending in a 1, and c_n, the number of decimal digit sequences (0–9) of length n with no consecutive 1's and ending in a 1.

4. t_n, the maximum (worst case scenario) number of long divisions required to find $\gcd(a, b)$ using the Euclidean Algorithm, if $a, b \leq n$.

   Background: The Euclidean Algorithm finds $\gcd(a, b)$ by repeated long division. In the worst case scenario, from a given starting $a, b \leq n$, after 2 long divisions, we get $a', b' \leq \lfloor n/2 \rfloor$ such that $\gcd(a', b') = \gcd(a, b)$. ($\lfloor x \rfloor$ is the greatest integer less than x, i.e., x rounded down.) We then apply the Euclidean algorithm to a', b'.

7.3: Find closed formulas for the following sequences given by linear recurrence relations.

1. $a_n = 2a_{n-1} + 3a_{n-2}$, $a_1 = 7$, $a_2 = 5$.
2. $a_n = a_{n-1} - a_{n-2}$, $a_1 = 1$, $a_2 = 2$.
3. $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$, $a_1 = 0$, $a_2 = 0$, $a_3 = 1$.

7.3: Find linear recurrence relations and closed formulas for the following sequences. You may use a computer algebra system (e.g., Wolfram Alpha) to find roots of polynomials, etc.

1. The number of sequences of English letters not containing the word HA.
2. The number of $2 \times n$ Lego walls of height 1 made from red 2×1 bricks and blue 2×4 bricks.
3. The number of tilings of an $n \times 1$ grid by white dominoes (2×1) and red and blue triominos (3×1).

7.4: Find closed formulas for the following sequences, using the following table.

Recurrence ($c \neq 1$)	Solution form
$a_n = ca_{n-1} + d$	$a_n = Ac^n + B_0$
$a_n = ca_{n-1} + dn + e$	$a_n = Ac^n + B_1 n + B_0$
$a_n = ca_{n-1} + dn^2 + en + f$	$a_n = Ac^n + B_2 n^2 + B_1 n + B_0$

1. $a_n = 3a_{n-1} + (n-5)$, $a_0 = 2$.
2. $a_n = 5a_{n-1} + n^2$, $a_0 = 0$.
3. $a_n = 2a_{n-1} + (3n-1)$, $a_0 = 4$.

7.5: For each a_n, find the generating function $g(x) = \sum_{n=0}^{\infty} a_n x^n$.

1. $a_n = a_{n-1} + (2n - 1)$, $a_0 = 0$.
2. $a_n = 5a_{n-1} + 13$, $a_0 = 1$.

8.1: Suppose A, B, C are finite sets.

1. The goal of this problem is to find a formula for $|A \cup B|$.
   (a) Suppose we try the formula $|A| + |B|$ first. How many times does $|A| + |B|$ count each element of $A \cap B$? (Try drawing a Venn diagram to see.)
   (b) Adjust $|A| + |B|$ with a term involving $|A \cap B|$ to get a correct formula of the form $|A \cup B| = |A| + |B| + ??$.
2. The goal of this problem is to find a formula for $|A \cup B \cup C|$.
   (a) How many times does the formula $|A| + |B| + |C|$ count each element of $A \cap B$, $A \cap C$, and $B \cap C$? How about $A \cap B \cap C$?
   (b) Adjust $|A| + |B| + |C|$ with terms involving $|A \cap B|$, etc., to get a formula that counts all elements of $A \cap B$, etc., exactly once. How many times does your formula count elements of $A \cap B \cap C$?
   (c) Now adjust your formula from 2b to get a correct formula of the form $|A \cup B \cup C| = |A| + |B| + |C| + ??$.

8.1:

1. A school has 200 students with 85 students taking each of three subjects: trigonometry, probability, and basket-weaving. There are 30 students taking any given pair of these subjects, and 15 students taking all three subjects.
   (a) How many students are taking none of these three subjects?
   (b) How many students are taking only probability?
2. How many arrangements are there of TAMELY are there with
   (a) T before A?
   (b) T before A and A before M?
   (c) Either T before A, or A before M, or M before E?

   Note: "Before" means *anywhere* before.
3. In a class of 30 students, 20 take Latin, 14 take Greek, and 10 take Hebrew. No student takes all three languages and eight students take no language.
   (a) How many students are taking two languages?
   (b) How many students take Greek and Hebrew?

8.2: In each of these problems, we want to enumerate a set of objects that avoid multiple conditions. To start each problem:

- Think: What is **one** condition that we want to avoid?
- The A_1 in your problem then becomes the set of all objects that **meet** one of the conditions you want to avoid, and similarly for all A_i.
- Then enumerate $A_1 \cap A_2$, and so on, and apply inclusion-exclusion to enumerate $X - (A_1 \cup \cdots \cup A_n)$.

1. How many arrangements are there of AAABBBCCCDDDEEE without three consecutive letters the same?
2. How many ways are there to distribute 29 identical balls into 9 distinct boxes with at most 7 balls in any of the first 4 boxes?
   (a) Solve this problem with inclusion-exclusion.
   (b) Solve this problem again using generating functions.
3. How many ways are there for a child to take 13 pieces of candy, choosing from among lollipops, gumballs, peanut butter cups, mega-gummy worms, and peppermint patties, if the child does not take exactly 3 pieces of any type of candy?
   (a) Solve this problem with inclusion-exclusion.
   (b) How would you solve this problem with generating functions?

Bonus problem: Anh, Betty, Chuck, Darby, and Earl are violinists, and Amy, Bright, Callie, Diego, and Edwin are pianists. Each violinist usually plays with the pianist whose name starts with the same letter. Suppose they now randomly re-choose violin-piano pairs.

- What is the probability that Anh will be re-paired with Amy?
- What is the probability that **at least one** original pair will be randomly re-paired (Anh/Amy, Betty/Bright, etc.)? What is the probability that **no** original pair will be re-paired?
- Generalize to n violinists and n pianists.