Math 131B, Mon Dec 07

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- PS11 due tonight.
- All revisions due **Fri Dec 18**.
- Final exam review, **Thu Dec 10, 12:30–2:30pm.**
- **FINAL EXAM, MON DEC 14, 9:45am–noon.**

Chapters 2 and 3

Chapter 1-3 will be less emphasized on the final.

- 2.1–2.2: $\mathbb{R}$ and $\mathbb{C}$
- 2.3–2.5: Metrics, limits in metric spaces, completeness
- 3.1: Continuity
- 3.2: Differentiation
- 3.3–3.4: Definition of the Riemann integral
- 3.5: FTC and parts

Sections 4.1–4.3

- 4.1: Infinite series: Basic defns and tests (comparison, ratio)
- 4.2: Sequences and series of functions: The six NOs
- 4.3: Uniform convergence
- 4.3: The YESes
- 4.3: Weierstrass M-test

If $|g_n(x)| \leq M_n$, $\sum M_n$ convs

$\Rightarrow \sum g_n(x)$ convs abs & <u>unif</u>.

Sections 4.4–4.6

- 4.4: Power series (radius of convergence)
- 4.5: Complex exponential and trig functions
- 4.6: Properties of complex exponentials: Unit circle, integrals

$e^{2\pi i x}$

$\int x^k \overline{e_n(x)}\, dx$

Chapter 5

L^1, L^2, L^∞

- 5.1: Different ways to measure the distance between functions
- 5.2: Function spaces and metrics
- 5.3: Dot products and geometry

$L^2(S^1)$ like $\mathbb{C}^N$

$\langle f, g\rangle = \int_0^1 f\bar{g}\,dx$ $\qquad$ $\langle v, w\rangle = v\cdot\bar{w}$

$\{e_n(x)\}$ $\qquad$ $\{e_n\}$

Sections 6.1, 6.2, 6.4

$\hat{f}(n)$
$= \langle f, e_n \rangle$

$f \in C^0(S^1)$

$= \int_0^1 f(x)\overline{e_n(x)}\,dx$

- 6.1: The Nth Fourier polynomial f_N
- 6.2: The Fourier series of f
- 6.4: The not-so-great C^2 convergence theorem
- 6.4: The $\widehat{f'}(n)$ formula

(replaced later)

$f \in C^1(S^1)$

$\widehat{\frac{df}{dx}}(n) = (2\pi i n)\hat{f}(n)$

Sections 7.1–7.3

$L^2(S^1)$ as IP space

- 7.1: Inner products and inner product spaces
- 7.1: Example: IP on $C^0(S^1)$
- 7.1: Projection, Cauchy-Schwarz, triangle
- 7.2: Norms and normed spaces
- 7.2: Relationships among L^2, L^1, uniform, and pointwise convergence
- 7.3: Orthogonal sets and projection ← why L^2 special
- 7.3: Best Approximation and Always Better Theorems

$$\|f\| = \left(\int_X |f|^2\,dx\right)^{1/2} \qquad \|f\|_1 = \int_X |f|\,dx$$

Sections 7.4–7.6

- (7.4: Sets of measure zero)
- (7.5: The Lebesgue integral)
- 7.5: $L^1(X)$, $L^2(X)$, $X = \mathbb{R}, S^1$

$L^2(S^1)$ complete

- 7.6: Hilbert space defn and examples $L^2(S^1)$
- 7.6: Hilbert Space Absolute Convergence Thm

HSACT

- 7.6: Isomorphism theorem

$\{e_n\}$ orthonormal basis for $\mathcal{H}$

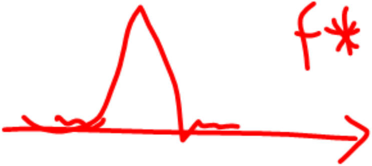
$\Rightarrow \|f\|^2 = \sum |\hat{f}(n)|^2$

Sections 8.1–8.3

- 8.1: Recap/reboot of $\hat{f}(n)$ on $L^2(S^1)$
- 8.2: Defn of convolution
- 8.2: Properties of convolution (esp. $\widehat{f * g}$)

$$\widehat{f*g}(n) = \hat{f}(n)\hat{g}(n)$$

- 8.3: Defn of Dirac kernels
- 8.3: Dirichlet and Fejér kernels



$f * F_N$ = Nth Cesàro sum

Sections 8.4, 8.5.3

$f \in L^2(S^1)$

$\Rightarrow f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n(x)$ $\boxed{\text{in } L^2}$

- 8.4: The $L^2(S^1)$ Inversion Theorem
- 8.4: The $C^1(S^1)$ Pointwise Inversion Theorem
- 8.5.3: Application to Riemann Zeta Function

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

If $f \in C^1(S^1)$, works abs & unif.

Can compute $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Sections 4.7, 4.8, 12.1: Fourier Transform

Definition (Schwartz space)

$\mathcal{S}(\mathbb{R})$ is the space of all $f : \mathbb{R} \to \mathbb{C}$ such that for all $k \geq 0$, the kth derivative $f^{(k)}(x)$ of f exists for all $x \in \mathbb{R}$ and is rapidly decaying.

Recall: Parts, Fubini, differentiation under the integral sign, all on $\mathbb{R}$.

$f \in S(\mathbb{R})$

Definition

For $f \in \mathcal{S}(\mathbb{R})$, define the **Fourier transform** of f to be the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ given by

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \gamma x} \, dx$$

for any $\gamma \in \mathbb{R}$.

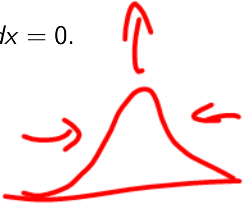
Note that because we now assume $f \in \mathcal{S}(\mathbb{R})$, integral definitely converges.

Section 12.2: Convolution and Dirac kernel

- Convolution $f * g : \mathbb{R} \to \mathbb{C}$ defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt.$$

- Dirac kernel $K_t : \mathbb{R} \to \mathbb{R}$ ($t \in \mathbb{R}$, $t > 0$):
  - For all $t > 0$ and all $x \in \mathbb{R}$, $K_t(x) \geq 0$;
  - For all $t > 0$, $\int_{-\infty}^{\infty} K_t(x)\,dx = 1$; and
  - For fixed $\eta > 0$, we have $\lim_{t\to 0^+} \int_{|x|\geq\eta} K_t(x)\,dx = 0$.
- Thm: $\lim_{t\to 0^+} (f * K_t)(x) = f(x)$.
- Example of a Dirac kernel: Gauss kernel



$(t \to 0^+)$

$$G_t(x) = \frac{1}{t}\exp\left(\frac{-\pi x^2}{t^2}\right).$$

Section 12.3: Properties of Fourier transform

Function (in x)	Fourier transform (in γ)
$f(x+a)$	$e^{2\pi i a\gamma}\hat{f}(\gamma)$
$e^{2\pi i a x}f(x)$	$\hat{f}(\gamma - a)$
$f(-x)$	$\hat{f}(-\gamma)$

$\widehat{f*g} = \hat{f}\hat{g}$

etc.

Theorem

If $f, g \in \mathcal{S}(\mathbb{R})$, then $\displaystyle\int_{-\infty}^{\infty} \hat{f}(x)g(x)\,dx = \int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx.$

Theorem

The Fourier transform of $f(x) = e^{-\pi x^2}$ is $\hat{f}(\gamma) = e^{-\pi\gamma^2}$. In other words, f is its own Fourier transform, or $U(f) = f$.

More generally, for $t > 0$, let $G_t(x) = \dfrac{1}{t}\exp\left(\dfrac{-\pi x^2}{t^2}\right)$ be the Gauss kernel. Then

$$\hat{G}_t(\gamma) = e^{-\pi t^2\gamma^2}, \qquad U(U(G_t)) = \hat{\hat{G}}_t = G_t.$$

Section 12.4: Inversion and Isomorphism Theorems

Theorem

For $f \in \mathcal{S}(\mathbb{R})$, we have that

$$\hat{\hat{f}}(x) = \int_{-\infty}^{\infty} \hat{f}(\gamma) e^{-2\pi i \gamma x}\, d\gamma = f(-x).$$

I.e., $f(x) = \int_{-\infty}^{\infty} \hat{f}(\gamma) e^{2\pi i \gamma x} d\gamma$

Theorem

For $f, g \in \mathcal{S}(\mathbb{R})$, we have that $\langle f, g \rangle = \left\langle \hat{f}, \hat{g} \right\rangle$. In particular, $\|f\| = \left\|\hat{f}\right\|$.

Compare isomorphism theorem for $L^2(S^1)$:

Theorem

For $f, g \in L^2(S^1)$, we have that

$$\langle f, g \rangle = \sum_{n \in \mathbb{Z}} \hat{f}(n) \overline{\hat{g}(n)}. = \langle \hat{f}, \hat{g} \rangle$$

dot prod!

How to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$f(x) = x$

$-\frac{1}{2} \leq x < \frac{1}{2}$

$\hat{f}(0) = 0$

$\hat{f}(n) = -\frac{(-1)^2}{2\pi i n}$

B/c $\|f\|^2 = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2$

$\{e_n(x)\}$ orthonormal basis for $L^2(S^1)$,

$$\|f\|^2 = \int_{S^1} |f(x)|^2 \, dx$$

LHS

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \, dx = I$$

$$\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 = \sum_{n \neq 0} \left| - \frac{(-1)^2}{2\pi i n} \right|^2$$

$$=$$

$$= c \sum \frac{1}{n^2}$$

$\zeta(4)$ sim, but more.