Math 131B, Wed Nov 04

We have class on Mon Nov 09, but will not meet on Wed Nov 11.
We *will* have a problem session on Fri Nov 13.

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 8.3. Reading for Mon: 8.4.
- PS08 due today; PS09 outline due Mon.
- Problem session, Fri Nov 06, 10:00–noon on Zoom.

$\boxed{\text{PS08 \& 09}}$

Recap: Two tools

(for proving Fundamental Theorem of Fourier Series and beyond)

First is convolutions:

Definition

For $f, g \in C^0(S^1)$, the **convolution** $f * g : S^1 \to \mathbb{C}$ is defined by the formula

$$(f * g)(x) = \int_0^1 f(x-t)g(t)\,dt.$$

Second goes hand in hand with convolutions: **Dirac kernels**.

Start of Prob 8.2.7

$$(f*g)(x) = \int_0^1 \underbrace{f(x-t)}_{\text{const w.r.t } t}\, g(t)\,dt$$

Thm: $\widehat{f*g}(n) = \hat{f}(n)\hat{g}(n)$.

$$\widehat{f*g}(n) = \underbrace{\int_0^1 (f*g)(x)\,\overline{e_n(x)}\,dx}_{\text{IP } \langle f*g, e_n \rangle}$$

$$= \int_0^1 \underbrace{\left(\int_0^1 f(x-t)\,g(t)\,dt\right)}_{\text{defn of } f*g} \overline{e_n(x)}\,dx$$

Tools: Fubini, substitution, periodicity

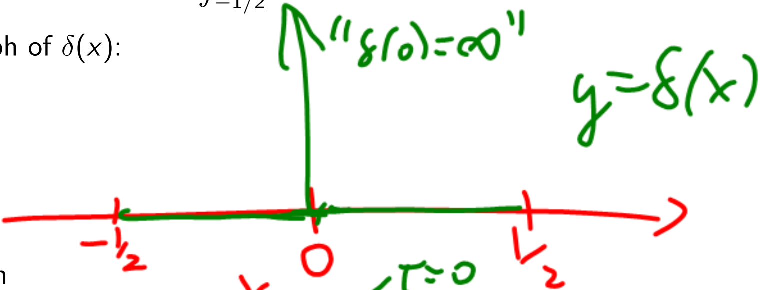
Wishful thinking: What is a Dirac kernel?

BEGIN WISHFUL THINKING

Suppose there existed a **Dirac delta function** $\delta(x)$ such that for $f \in C^0(S^1)$,

$$\int_{-1/2}^{1/2} \delta(x) f(x)\, dx = f(0).$$

Graph of $\delta(x)$:



Then

$t \approx 0$

$$(f * \delta)(x) = \int_{-1/2}^{1/2} f(x-t)\,\delta(t)\,dt = f(x-0) = f(x)$$

Still wishful thinking: Kernels$\to \delta$

↑ so δ = id of $*$

K_n: 

To prove $\lim_{n\to\infty} f_n = f$, suppose we can find K_n such that $f * K_n = f_n$ and $\lim_{n\to\infty} K_n = \delta$. Then: (wishfully thinking lim commutes w/ *)

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} f * K_n = f * \lim_{n\to\infty} K_n = f * \delta = f$$

END WISHFUL THINKING

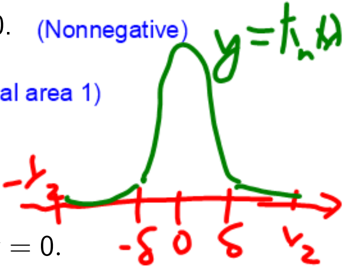
We now use that idea to motivate a rigorous defn.

Dirac kernels

Definition

A **Dirac kernel** on S^1 is a sequence of continuous functions $K_n : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}$ such that

1. For all n and all $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $K_n(x) \geq 0$. (Nonnegative)
2. For all n, $\displaystyle\int_{-1/2}^{1/2} K_n(x)\,dx = 1$; and (Total area 1)
3. For any fixed $\delta > 0$, we have



(Concentrated at 0)

$$\lim_{n\to\infty} \int_{\delta \leq |x| \leq \frac{1}{2}} K_n(x)\,dx = 0.$$

$\int_{-\frac{1}{2}}^{-\delta} + \int_{\delta}^{\frac{1}{2}}$

I.e., for $\delta > 0$ and $\epsilon > 0$, there exists some $N(\epsilon)$ such that for $n > N(\epsilon)$, we have

$$1 - \epsilon < \int_{-\delta}^{\delta} K_n(x)\,dx \leq 1.$$

A near-example and an example: Dirichlet kernel and Fejér kernel

Example

The **Dirichlet kernel** $\{D_N \mid N \geq 0\}$ is

$$D_N(x) = \sum_{n=-N}^{N} e_n(x).$$

(We'll see: $f * D_N = f_N$)

Example

The **Fejér kernel** $\{F_N \mid N \geq 1\}$ is

$$F_N(x) = \frac{D_0(x) + \cdots + D_{N-1}(x)}{N}.$$

$f * F_N = \frac{f_0 + \cdots + f_{N-1}}{N}$

Dirichlet kernel looks good at first, but as it turns out, only Fejér has the analytic properties we need to be a Dirac kernel.
To Maple!

Algebraic properties of Dirichlet and Fejér kernels

Theorem

For $f \in C^0(S^1)$, $f * D_N = f_N$, the Nth Fourier polynomial of f, and

$$(f * F_N)(x) = \frac{f_0(x) + \cdots + f_{N-1}(x)}{N} = s_N(x)$$

the average of the Fourier polynomials $f_0, \ldots, f_{N-1}$.

Proof: PS09.

Definition

The above sum $s_N(x) = (f * F_N)(x)$ is called the Nth **Cesàro sum** of the Fourier series of f.

Avgs make conv slower but smoother

Handy and remarkable summation formulas

Lemma

For $x \in S^1$, $n \geq 0$, and $N \geq 1$, we have that

$$D_n(x) = \begin{cases} \dfrac{\sin((2n+1)\pi x)}{\sin(\pi x)} & \text{if } x \neq 0, \\ 2n+1 & \text{if } x = 0, \end{cases}$$

$$F_N(x) = \begin{cases} \dfrac{1}{N} \dfrac{\sin^2(N\pi x)}{\sin^2(\pi x)} & \text{if } x \neq 0, \\ N & \text{if } x = 0. \end{cases}$$

$\boxed{-\frac{1}{2} \leq x \leq \frac{1}{2}}$

Proof uses clever arguments with geometric series, but we mostly just care that the result is true.

$q = e^{2\pi i x} \quad e_n(x) = q^n$

$D_N(x) = e_{-N}(x) + \cdots + e_0(x) + \cdots + e_N(x)$

$= q^{-N} + q^{-(N-1)} + \cdots + q^0 + \cdots + q^{N-1} + q^N$

$= q^{-N}(1 + q + \cdots + q^{2N})$ ← geom series

$= q^{-N}\left(\frac{1 - q^{2N+1}}{1-q}\right)$ (or something like that)

The Fejér kernel F_N is a Dirac kernel

3 things to check:

1. (Nonnegative) $F_n(x) \geq 0$ for all x.

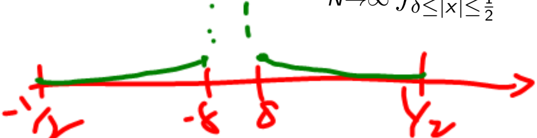
2. (Area 1) PS09 shows: For all N,

$$\int_{-1/2}^{1/2} F_N(x)\, dx = 1.$$

3. (Concentrated at 0) PS09 shows: For any $\delta > 0$,

(whatever)

$$\lim_{N\to\infty} \int_{\delta \leq |x| \leq \frac{1}{2}} F_N(x)\, dx = 0.$$



Next steps (8.4)

We next prove:

(ex: F_N)

Theorem

If $\{K_N\}$ is a Dirac kernel, and $f \in C^0(S^1)$, then

$$\lim_{N\to\infty} (f * K_N)(x) = f(x),$$

where convergence is uniform on S^1.

This shows that the Césaro sums of the Fourier polynomials of f converge uniformly to f.
(Note: That's not true of the Fourier polynomials themselves, which might diverge on, say, some uncountable set of measure zero.)

(the Hilbert space stuff)

We will then add that fact to the theoretical framework that we developed in Ch. 7 to show that the Fourier polynomials of f converge to f **in the L^2 metric**.