Math 131B, Wed Oct 21

7.4: This is where it gets weird for a bit.

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 7.4. Reading for Mon: 7.5.
- PS07 outline due **Mon** (not today).
- Problem session Fri Oct ~~16~~, 10:00–noon on Zoom.

23

Recap: Orthogonal sets and bases

$I = \mathbb{N}$ or $\mathbb{Z}$

or $\{1, \ldots, N\}$

Definition

Let V be an inner product space and let I be an index set. To say that $\mathcal{B} = \{u_i \mid i \in I\} \subset V$ is an **orthogonal set** means that for $i \neq j$, u_i and u_j are orthogonal (i.e., $\langle u_i, u_j \rangle = 0$).
To say that $\mathcal{B} = \{e_i \mid i \in I\} \subset V$ is an **orthonormal set** means that $\mathcal{B}$ is an orthogonal set and also, for every $i \in I$, $\langle e_i, e_i \rangle = 1$.

THE example: For $V = C^0(S^1)$ with usual L^2 IP and $e_n(x) = e^{2\pi inx}$, $\{e_n(x) \mid n \in \mathbb{Z}\}$ is orthonormal.

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}\,dx$$

Generalized Fourier polynomials and series THE ex!

V an IP space, $\{u_n\}$ an orthogonal set of nonzero vectors, $f \in V$.

nth generalized Fourier coefficient: $(\|e_n\|=1)$

$$\hat{f}(n) = \frac{\langle f, u_n \rangle}{\langle u_n, u_n \rangle} = \frac{\langle f, u_n \rangle}{\|u_n\|^2}. = \langle f, e_n \rangle = \hat{f}(n)$$

Nth generalized Fourier polynomial:

$\mathcal{B} = \{u_1 \cdots u_N\}$

$$\operatorname{proj}_{\mathcal{B}} f = \sum_{n=1}^{N} \hat{f}(n) u_n = \sum_{n=1}^{N} \frac{\langle f, u_n \rangle}{\langle u_n, u_n \rangle} u_n = f_N$$

Generalized Fourier series of f:

$\mathcal{B} = \{u_1, \ldots\}$

$$\lim_{N \to \infty} \sum_{n=1}^{N} \hat{f}(n) u_n = \sum_{n=1}^{\infty} \hat{f}(n) u_n$$

or

$\mathcal{B} = \{\ldots u_{-1}, u_0, u_1, \ldots\}$

$$\sum_{n \in \mathbb{Z}} \hat{f}(n) u_n. = FS \text{ of } f$$

Note: If $\{u_1, \dots, u_N\}$ orthog, Pythag $\Rightarrow$ orthog sum

$$\left\| \sum_{n=1}^{N} c_n u_n \right\|^2 = \sum_{n=1}^{N} \| c_n u_n \|^2$$

$$= \sum_{n=1}^{N} |c_n|^2 \| u_n \|^2$$

Best Approximation Theorem

This gives the meaning of proj_B (f).

Theorem (Best Approximation Theorem)

V a IP space, $\mathcal{B} = \{u_1, \ldots, u_N\}$ be an orthogonal set of nonzero vectors in V, $f \in V$.

1. *For $1 \le n \le N$, the vector $f - \mathrm{proj}_{\mathcal{B}} f$ is orthogonal to u_n.*
2. *For any $c_1, \ldots, c_N \in \mathbb{C}$, we have*

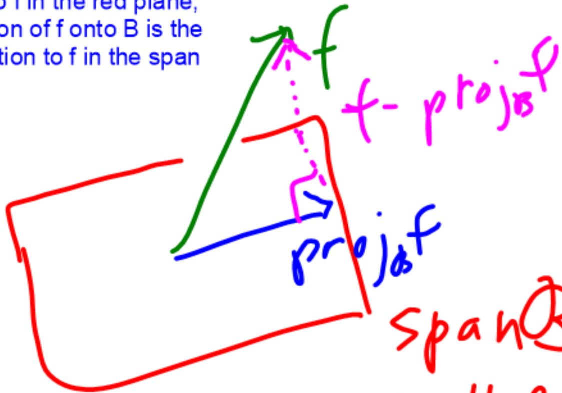
$$\left\| f - \sum_{n=1}^{N} c_n u_n \right\|^2 = \sum_{n=1}^{N} \left| \hat{f}(n) - c_n \right|^2 \langle u_n, u_n \rangle + \| f - \mathrm{proj}_{\mathcal{B}} f \|^2 .$$

how close to f?

3. *The vector $\mathrm{proj}_{\mathcal{B}} f$ is the unique element in the span of $\mathcal{B}$ that is closest to f in the L^2 metric.*
4. *(Bessel's inequality)*

$$\|\mathrm{proj}_{\mathcal{B}} f\|^{2} \overbrace{= \sum_{n=1}^{N} \left| \hat{f}(n) \right|^2 \langle u_n, u_n \rangle}^{\text{Pythag.}} \le \|f\|^2 .$$

(3) says that the blue vector is the closest vector to f in the red plane; i.e., the projection of f onto B is the best approximation to f in the span of B.



span $\mathcal{B}$ = all l.c. of $\{u_1, \dots, u_N\}$

(4) $\|\text{proj}\| \le \|f\|$

Proof of (1) and (2) on PS07.

Proof of (3) and (4):

arb l.c. of u_n

(2)

$$\left\| f - \sum_{n=1}^{N} c_n u_n \right\|^2 = \sum_{n=1}^{N} \left| \hat{f}(n) - c_n \right|^2 \langle u_n, u_n \rangle + \overbrace{\| f - \mathrm{proj}_{\mathcal{B}} f \|^2}^{\text{Ind of } c_n} .$$

Pf of (3): To choose c_n to minimize the LHS of above, choose c_n to make sum on RHS equal to 0, i.e., choose

$c_n = \hat{f}(n)$.

(4) Take $c_n = 0$.

$$\|f\|^2 = \sum_{n=1}^{N} |\hat{f}(n)|^2 \langle u_n, u_n \rangle + (\geq 0)$$

$$\|f\|^2 \geq \sum_{n=1}^{N} |\hat{f}(n)|^2 \langle u_n, u_n \rangle = \|\mathrm{proj}\|^2$$

Always Better Theorem

$\{u_1, u_2, \dots\}$

Corollary (Always Better Theorem)

Let V be an inner product space, and let $\mathcal{B} = \{u_n \mid n \in \mathbb{N}\}$ be an orthogonal set of nonzero vectors in v. Then for $f \in V$ and $1 \leq K \leq N$, we have that

$$\left\| f - \underbrace{\sum_{n=1}^{N} \hat{f}(n) u_n}_{\text{Nth FP (later)}} \right\| \leq \left\| f - \underbrace{\sum_{n=1}^{K} \hat{f}(n) u_n}_{\text{Kth FP}} \right\|.$$

L^2 size of error in approxs N and K, resp.

I.e., later approx always get better, or at least don't get worse.

Orthogonal and orthonormal bases

for V

Definition

V an IP space. To say that $\mathcal{B} = \{u_n \mid n \in \mathbb{N}\} \subset V$ is an **orthogonal basis** means that $\mathcal{B}$ is an orthogonal set of nonzero vectors and for any $f \in V$, the generalized Fourier series of f converges to f in the inner product metric. I.e., for $f \in V$,

I.e.: Any f is an "infinite linear combination" of u_n, where conv is in L^2.

$$\sum_{n=1}^{\infty} \hat{f}(n) u_n = \lim_{N \to \infty} \sum_{n=1}^{N} \hat{f}(n) u_n = f,$$

where convergence is in L^2. Two-sided orthogonal basis similar except $\sum_{n \in \mathbb{Z}}$. **Orthonormal basis** defined analogously, replacing "orthogonal set of nonzero vectors" with "orthonormal set."

THE main problem, reframed: Prove that $\{e_n\}$ is an orthonormal basis for $C^0(S^1)$. (Note that convergence is in L^2, not the same as pointwise or uniform.)

I.e.

(For $f \in C^0(S^1)$)

Prove $\sum_{n \in \mathbb{Z}} \hat{f}(n) e_n(x)$ convs to f in L^2

$\Updownarrow$

Prove $\{e_n \mid n \in \mathbb{Z}\}$ is orthonormal basis for $C^0(S^1)$.

Why the Lebesgue integral?

7.4-7.5

For an optimal theory of $\{e_n\}$ as an orthonormal basis, need to overcome the fact that $C^0(S^1)$ has "holes":

- It is possible to have a sequence of Riemann integrable functions whose pointwise limit is not Riemann integrable.
- If we look at the space $V = C^0([a, b])$ of continuous functions on a closed and bounded interval under the L^2 metric, we see that V is not complete as a metric space, just like $\mathbb{Q}$.

We can fill in those "holes" by defining what is known as the **Lebesgue integral**.

This is a way to extend Riemann integral to stranger kinds of functions.

The axiomatic approach

Instead of fully defining the Lebesgue integral, which takes a whole semester (Math 231A), we axiomatize its properties and assume it exists, much like we assumed that $\mathbb{R}$ exists.

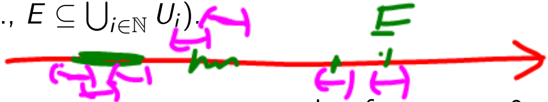
However, even to describe those desired properties, we need to understand one particular idea from measure theory: sets of **measure zero**.

Measure zero

(Think: What does it mean for a subset of R to have zero length?)

Definition

We define the **length** of an open interval (a, b) to be $\ell((a, b)) = b - a$. For $E \subseteq \mathbb{R}$, we define a **countable open cover** of E to be a countable collection $\{U_i\}$ of open intervals whose union contains E (i.e., $E \subseteq \bigcup_{i \in \mathbb{N}} U_i$).



Definition

To say that $E \subseteq \mathbb{R}$ has **measure zero** means that for any $\epsilon > 0$, there exists some open cover $\{U_i\}$ of E such that $\displaystyle\sum_{i=1}^{\infty} \ell(U_i) < \epsilon$.

ctble

Definition

For $X \subseteq \mathbb{R}$, to say that a statement is true **almost everywhere**, or **a.e.**, in X, means that the set of points in X where the statement does not hold has measure 0. **Almost all**, etc., defined similarly.

Example

$\forall \epsilon > 0$

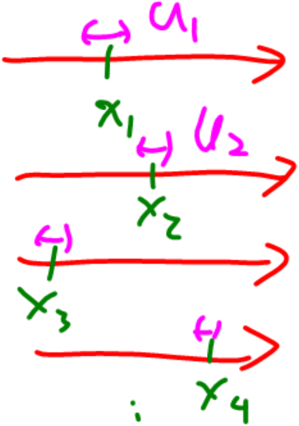
So E = rationals has measure zero.

Example-thm. If $E = \{x_i\}$ is a countable subset of $\mathbb{R}$, then E has measure zero.

ctble $\{u_i\}$ of E s.t. $\sum_{i=1}^{\infty} \ell(u_i) < \epsilon$.

Proof:

(1) $\epsilon > 0$



$\ell(u_1) = \frac{\epsilon}{4}$

$\ell(u_2) = \frac{\epsilon}{8}$

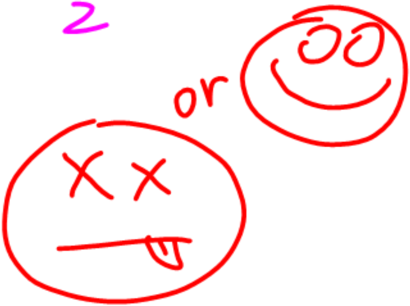
$\ell(u_3) = \frac{\epsilon}{16}$

$\ell(u_4) = \frac{\epsilon}{32}$

⋮

$$\ell(u_i) = \frac{\epsilon}{2^{i+1}}$$

$$\sum_{i=1}^{\infty} \ell(u_i) = \frac{\epsilon}{2} < \epsilon.$$



A technical lemma

Lemma

(A, B) open int in $\mathbb{R}$, $\{U_i\}$ countable collection of open int in (A, B). There exists a countable collection $\{V_j\}$ of bounded open ints s.t.:

1. *(Disjoint) For $j \neq k$, $V_j \cap V_k = \emptyset$;*
2. *(Union)* $\displaystyle\bigcup_{j=1}^{\infty} V_j = \bigcup_{i=1}^{\infty} U_i$*; and*
3. *(Shorter)* $\displaystyle\sum_{j=1}^{\infty} \ell(V_j) \leq \sum_{i=1}^{\infty} \ell(U_i)$.

Next time

Picture:

Typical of intro measure theory: "Obvious", but proof is complicated.

A set of measure zero can't contain an interval

Theorem

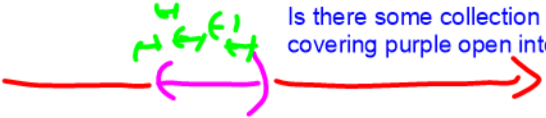
If E is a set of measure zero, and (a, b) is any open interval in $\mathbb{R}$, then (a, b) is not contained in E.

Proof:

E, meas 0

can't contain

Q Can you cover



Is there some collection of green open intervals covering purple open interval whose total length is less than any epsilon?

See: Banach-Tarski paradox on YouTube.

Continuous and equal a.e. means equal

Corollary

Suppose $X = [a, b]$ *or* $\mathbb{R}$ *and for some* $f, g : X \to \mathbb{C}$*, we have that* $f(x) = g(x)$ *for almost all* $x \in X$*. Then for* $c \in X$*, if* f *and* g *are continuous at* c*, then* $f(c) = g(c)$*.*

Proof: