Math 131B, ~~Mon Oct 12~~ Wed Oct 14

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 7.2. Reading for next Wed: 7.3.
- PS06 due today. ~~PS07 outline due in 1 week.~~
- **EXAM ~~1~~ 2 on Mon Oct 19.** Chs 4, 5, 6; 7.1
- Exam review Fri Oct 16, 10:00–noon on Zoom.

PS04-06

Recap of normed spaces $\boxed{7.2}$

Definition

V a fn space. A **norm** on V is $\|\cdot\| : V \to \mathbb{R}$ s.t.:

1. (Positive definite) For all $f \in V$, $\|f\| \geq 0$, and if $\|f\| = 0$, then $f = 0$.
2. (Absolute homogeneity) For all $f \in V$ and $a \in \mathbb{C}$, $\|af\| = |a| \, \|f\|$.
3. (Triangle inequality) For all $f, g \in V$, $\|f + g\| \leq \|f\| + \|g\|$.

A **normed space** is a fn space with a choice of norm.

So $d(f,g) = \|f-g\|$ is a metric

For $V = C^0(S^1)$, norms include:

L^1
$$\|f\|_1 = \int_{\cancel{X}\,S^1} |f(x)| \, dx$$

L^2
$$\underbrace{\|f\|}_{=\sqrt{\langle f,f\rangle}} = \|f\|_2 = \left(\int_{\cancel{X}\,S^1} |f(x)|^2 \, dx \right)^{1/2}$$

$$\|f\|_\infty = \sup \{ |f(x)| \mid x \in \cancel{X}\,S^1 \}$$

Different meanings of $f_n \to f$

Let $V = C^0([0,1])$, and consider f_n in V. Note that we have now defined $\lim_{n\to\infty} f_n = f$ in four different ways:

- *Pointwise convergence:* For every $x \in [0,1]$, $\lim_{n\to\infty} f_n(x) = f(x)$.
- *Uniform, or L^∞ convergence:* If $\|\cdot\|_\infty$ is the L^∞ norm on $C^0([0,1])$, then $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$, i.e., f_n converges uniformly to f on $[0,1]$.
- *L^1 convergence:* $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|\, dx = 0$.
- *L^2 convergence/inner product norm:* $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|^2\, dx = 0$.

The one for Fourier Series

Limit laws in a normed space

Because every normed space is a metric space, we can carry over the material we developed about limits and continuity in metric spaces.

Limit laws work in normed spaces pretty much as they work in $\mathbb{C}$.

Theorem V normed space

If f_n is a convergent sequence in V, then f_n is bounded. ($\|f_n\| < M$)

Theorem

Let f_n and g_n be sequences in V, and suppose that $\lim_{n\to\infty} f_n = f$, $\lim_{n\to\infty} g_n = g$, and $c \in \mathbb{C}$. Then we have that:

1. $\lim_{n\to\infty} cf_n = cf$; *and*
2. $\lim_{n\to\infty} (f_n + g_n) = f + g$.

Proofs are the same too. (replace $|\ |$ w/ $\|\ \|$)

Continuous functions between normed spaces

Definition

Let $T : V \to W$ be a function, where V and W are normed spaces (e.g., $W = \mathbb{C}$). For $g \in V$, to say that T is **continuous** at g means that one of the following conditions holds:

- (**Sequential continuity**) For every sequence f_n in V such that $\lim_{n\to\infty} f_n = g$, we have that $\lim_{n\to\infty} T(f_n) = T(g)$.
- (ϵ-δ **continuity**) For every $\epsilon > 0$, there exists some $\delta(\epsilon) > 0$ such that if $f \in V$ and $\|f - g\| < \delta(\epsilon)$, then $\|T(f) - T(g)\| < \epsilon$.

To say that T is **continuous on** V means that T is continuous at f for all $f \in V$.

Example/application

V an IP space and fix $g \in V$.

w.r.t. IP norm (L^2)

Theorem
$T_g : V \to \mathbb{F}$ defined by $T_g(f) = \langle f, g \rangle$ is continuous on V and similarly for $\overline{T}_g(f) = \langle g, f \rangle$.

Corollary (in L^2)
If $\sum_{n=1}^{\infty} f_n$ converges to f in IP norm, then:

$\langle \sum_{n=1}^{\infty} f_n, g \rangle$

I.e., with L^2 convergence, can pull out infinite sums, not just finite ones.

$$\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f_n, g \rangle , \qquad \langle g, f \rangle = \sum_{n=1}^{\infty} \langle g, f_n \rangle .$$

In particular, both RHS converge.

Later used for Fourier transform. Proof of both: PS07.

later in Ch.7 and

Outline of proof of Thm: Fix g in V. $T = T_g$

(A) $f_0 \in V$

(A) $\epsilon > 0$

Pick $\delta(\epsilon) = ?$

(A) $\|f - f_0\| < \delta(\epsilon)$

$\boxed{\text{LOTS OF STUFF}}$

(C) $\|T(f) - T(f_0)\| < \epsilon$

(C) $\forall f$ $\|f - f_0\| < \delta(\epsilon)$, then $\|T(f)$ . . .

(C) $\exists\, \delta(\epsilon) > 0$ s.t. if $\|f - f_0\| < \delta(\epsilon)$, then $\|T(f) - T(f_0)\| < \epsilon$

(C) T cont at f_0

(C) For $f_0 \in V$, T cont at f_0

(C) T cont on V

Cauchy sequences and Cauchy completeness in a normed space

V a normed space.

Definition

f_n be a sequence in V. To say that f_n is **Cauchy** means that for every $\epsilon > 0$, there exists some $N(\epsilon) \in \mathbb{R}$ such that if $n, k > N(\epsilon)$, then $\|f_n - f_k\| < \epsilon$.

I.e., the f_n get closer to each other as n -> infinity, but are not assumed to get closer to some limit f.

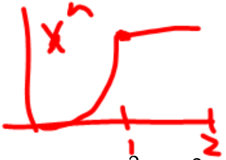
Definition

To say that a normed space V is **complete** means that any Cauchy sequence in V converges to some limit in V.

A Cauchy sequence in V whose L^2 limit is not in V

Let $V = C^0([0,2])$, and consider the following sequence in V:

$$f_n(x) = \begin{cases} x^n & \text{if } 0 \le x \le 1, \\ 1 & \text{if } x > 1. \end{cases}$$



A calculation shows that if $n, k > N(\epsilon) = \dfrac{2}{\epsilon^2}$, then $\|f_n - f_k\|^2 < \epsilon^2$, i.e., f_n is Cauchy. However, can show that the only possible L^2 limit of f_n is

in L^2

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x \ge 1, \end{cases}$$

which is not continuous, and therefore not in V. So V is not complete.

So V is not complete because we have to go outside V to find a limit of a particular Cauchy sequence.

This is like how Q (the field of the rationals) is not complete, because to find the limit of the Cauchy sequence:

3, 3.1, 3.14, 3.141, 3.1415, 3.14159,.....

we have to leave Q to get the limit of pi.

The upshot

How can we make $V = C^0([0,2])$ into a complete space?

- Could try to "plug the holes" in V by adding the limits of sequences like above f_n.
- But this makes more sequences of functions possible, which create new holes to plug. The Lebesgue int functions are enough to plug the holes.
- Process continues until we end up with functions that are not even (Riemann) integrable on $[0,2]$; instead, they are **Lebesgue integrable**. (More precisely, they are $f : [0,2] \to \mathbb{C}$ such that $|f(x)|^2$ is Lebesgue integrable on $[0,2]$.)
- Compare: The real numbers $\mathbb{R}$ are precisely what you get when you try to "plug the holes" in the rational numbers $\mathbb{Q}$. Note that in 131A, we didn't prove that you could do this; we just axiomatically assumed you could do it.
- Later (Sec 7.4–7.5), we will similarly assume axioms that allow you to plug the holes in V. (Math 231A then actually proves this is possible, without additional assumptions.)

Orthogonal sets and bases

$\boxed{7.3}$

V fn space

$\langle \, , \, \rangle$

$I = \mathbb{N}$ or $\mathbb{Z}$ or $\{1, \ldots, n\}$

Definition

Let V be an inner product space and let I be an index set. To say that $\mathcal{B} = \{u_i \mid i \in I\} \subset V$ is an **orthogonal set** means that for $i \neq j$, u_i and u_j are orthogonal (i.e., $\langle u_i, u_j \rangle = 0$).
To say that $\mathcal{B} = \{e_i \mid i \in I\} \subset V$ is an **orthonormal set** means that $\mathcal{B}$ is an orthogonal set and also, for every $i \in I$, $\langle e_i, e_i \rangle = 1$.

For us: We do Fourier series in terms of e_n(x).
This generalized theory also covers Fourier series in terms of:
- sin(2\pi n x), cos(2\pi n x)
- polynomials
- polynomials * Gaussians (version used for quantum mechanics)
- wavelets

(etc.) All of those are more naturally indexed by N than by Z.

THE example

Consider $C^0(S^1)$ with inner product

$$\langle f, g \rangle = \int_{S^1} f(x)\overline{g(x)}\,dx$$

Let $e_n(x) = e^{2\pi i n x}$, and recall:

$$\langle e_n, e_k \rangle = \int_0^1 e_n(x)\overline{e_k(x)}\,dx$$

$$= \begin{cases} 0 & n \neq k \\ 1 & n = k \end{cases}$$

So $\{e_n(x) \mid n \in \mathbb{Z}\}$ is orthonormal.

Generalized Fourier polynomials

Goal of the semester: Show {e_n(x)} is a "basis" for fns on S^1.

V an IP space, I an index set.

Definition

WTS that B is a "basis" for V.

Let $\mathcal{B} = \{u_n \mid n \in I\}$ be an orthogonal set of nonzero vectors in V. For $f \in V$ and $n \in I$, we define the nth **generalized Fourier coefficient of f with respect to $\mathcal{B}$** to be

$$\hat{f}(n) = \frac{\langle f, u_n \rangle}{\langle u_n, u_n \rangle} = \frac{\langle f, u_n \rangle}{\|u_n\|^2}.$$

If $\|u_n\| = 1$: $= \langle f, u_n \rangle$

If $\mathcal{B} = \{u_1, \ldots, u_N\}$, then we also define

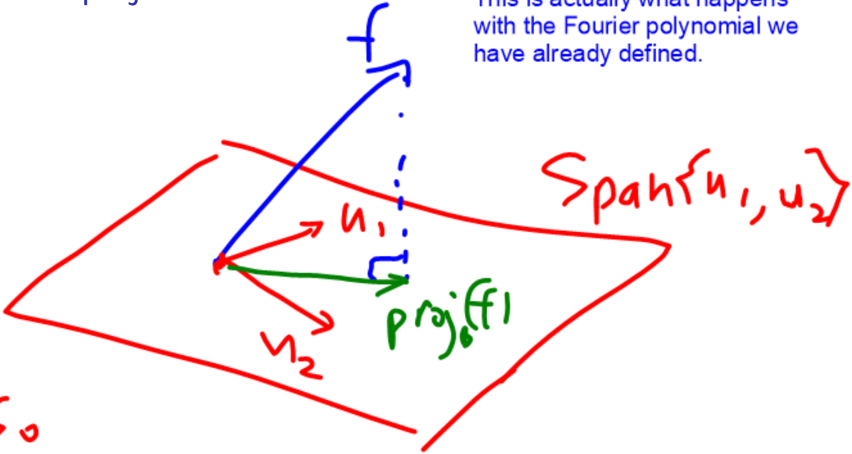
$$\operatorname{proj}_{\mathcal{B}} f = \sum_{n=1}^{N} \hat{f}(n) u_n = \sum_{n=1}^{N} \frac{\langle f, u_n \rangle}{\langle u_n, u_n \rangle} u_n$$

to be the **projection of f onto the span of $\mathcal{B}$**.

like $\sum_{n=-N}^{N} \hat{f}(n) e_n(x)$

Picture of projection

This is actually what happens with the Fourier polynomial we have already defined.



So

$$f = \underbrace{\mathrm{proj}_{\mathcal{B}}(f)}_{\text{in Span}\,\mathcal{B}} + (\text{stuff} \perp \text{to Span}\,\mathcal{B})$$

Generalized Fourier series

Let V an IP space, $\mathcal{B} = \{u_i \mid i \in \mathbb{N}\}$ an orthogonal set of nonzero vectors in V.

Definition

We define

$$f \sim \lim_{N\to\infty} \sum_{n=1}^{N} \hat{f}(n) u_n = \sum_{n=1}^{\infty} \hat{f}(n) u_n$$

to be the **generalized Fourier series of f with respect to $\mathcal{B}$**.

For $\mathcal{B} = \{u_i \mid i \in \mathbb{Z}\}$, we analogously have

$$f \sim \sum_{n\in\mathbb{Z}} \hat{f}(n) u_n.$$

Back to THE example

Take $V = C^0(S^1)$ with the L^2 inner product. Let

$$\mathcal{B}_N = \{e_0, e_1, e_{-1}, e_2, e_{-2}, \dots, e_N, e_{-N}\}.$$

Then:

- nth Fourier coefficient $\hat{f}(n)$ is exactly $\hat{f}(n) = \langle f, e_n \rangle$, as before.
- Projection of f onto the span of $\mathcal{B}_N$ is Nth Fourier polynomial of f.
- Generalized Fourier series with respect to $\mathcal{B} = \{e_0, e_1, e_{-1}, e_2, e_{-2}, \dots\}$ is usual Fourier series of f.

So why the abstraction?

- Includes other examples, like Fourier series with sines and cosines. (Not just a theory of one example!)
- Abstraction highlights what's important geometrically, as we'll see soon. . . .