Math 131B, Mon Oct 12

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today and Wed: 7.2.
- PS06 due Wed.
- **EXAM ~~1~~** 2 **in one week.** on PS04–06
- Exam review Fri Oct 16, 10:00–noon on Zoom.

Recap of IP spaces

Definition

An **inner product** on V is $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ s.t. for $f, g, h \in V$ and $a, b \in \mathbb{C}$,

$\langle f, ag + bh \rangle = \bar{a} \langle f, g \rangle + \bar{b} \langle f, h \rangle$

1. $\langle af + bg, h \rangle = a \langle f, h \rangle + b \langle g, h \rangle$
2. $\langle g, f \rangle = \overline{\langle f, g \rangle}$
3. $\langle f, f \rangle \geq 0$, and if $\langle f, f \rangle = 0$, then $f = 0$.

Definition

For $f \in V$, we define the **norm** of f to be $\|f\| = \sqrt{\langle f, f \rangle}$.

THE

Example

Let $X = [a, b]$ or S^1, and let $V = C^0(X)$. Then for $f, g \in V$,

$$\langle f, g \rangle = \int_X f(x) \overline{g(x)} \, dx$$

is an IP on V.

$a, b \in \mathbb{C}$ $\qquad \overline{\left(\frac{a}{b}\right)} = \frac{\bar{a}}{\bar{b}}$

$$\overline{ab} = \bar{a}\,\bar{b}$$

$$\overline{\left(\frac{i}{i}\right)} = \frac{-i}{-i} \quad \checkmark$$

$$\overline{a+b} = \bar{a} + \bar{b}$$

Complex conjugation is a ring automorphism of C.

$$\langle f-g, g\rangle$$

You can distribute differences
just like you distribute sums.

$$= \langle f, g\rangle + \langle -g, g\rangle$$

$$= \langle f, g\rangle - \langle g, g\rangle$$

Orthogonality and projection

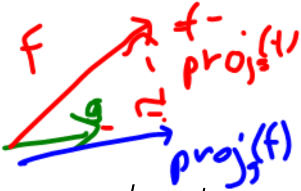
Definition

Let V be an inner product space. For $f, g \in V$, to say that f is **orthogonal** to g means that $\langle f, g \rangle = 0$.

Definition

Let V be an inner product space, and $g \neq 0$ in V. For $f \in V$, we define the **projection of** f **onto** $g$to be

$$\mathrm{proj}_g(f) = \frac{\langle f, g \rangle}{\langle g, g \rangle} g.$$



Theorem

Let V be an inner product space, and let g be a nonzero element of V. For $f \in V$, we have:

$$\left\langle \mathrm{proj}_g(f), g \right\rangle = \langle f, g \rangle ,$$
$$\left\langle f - \mathrm{proj}_g(f), \mathrm{proj}_g(f) \right\rangle = 0,$$
$$\left\| \mathrm{proj}_g(f) \right\| \leq \| f \| .$$

$$\langle g, g\rangle = \|g\|^2$$

$$\left|\frac{\langle f, g\rangle}{\langle g, g\rangle}\right| =$$

The extent to which f is pointed in the direction of g. If this is 0, then f is orthogonal to g, i.e., is pointed in a direction unrelated to g.

More precisely: As we'll see, if g is a coordinate vector for a set of coords for V,

$\frac{\langle f, g\rangle}{\langle g, g\rangle}$ is g-coord of f.

Cauchy-Schwarz and triangle

Theorem

V an IP space. For $f, g \in V$, we have:

1. *(Cauchy-Schwarz inequality)* $|\langle f, g\rangle| \leq \|f\| \, \|g\|$; *and*
2. *(Triangle inequality)* $\|f + g\| \leq \|f\| + \|g\|$.

Proof of C-S: First show: $|\langle f, g\rangle| = \left\|\text{proj}_g(f)\right\| \, \|g\|$.

$$|\langle f, g\rangle| = |\langle \text{proj}_g(f), g\rangle|$$

$$= \left|\left\langle \frac{\langle f, g\rangle}{\langle g, g\rangle} g, g \right\rangle\right|$$

pull constant out of IP

$$= \left|\frac{\langle f, g\rangle}{\langle g, g\rangle} \langle g, g\rangle\right| \quad \text{b/c}$$

$$= \left|\frac{\langle f,g\rangle}{\langle g,g\rangle}\right| \|g\|^2$$

$\langle g,g\rangle = \|g\|^2$

$$= \underbrace{\left|\frac{\langle f,g\rangle}{\langle g,g\rangle}\right| \|g\|}_{\|\mathrm{proj}_g(f)\|} \cdot \|g\|$$

b/c $\|ag\| = |a|\,\|g\|$

$$= \|\mathrm{proj}_g(f)\|\,\|g\|$$

b/c proj shrinks.

$$\leq \|f\|\,\|g\|.$$

Proof of triangle inequality

WTS: $\|f+g\| \leq \|f\| + \|g\|$

ETS this is >= 0

$$(\|f\|+\|g\|)^2 - \|f+g\|^2$$

$$= \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 - \langle f+g, f+g\rangle$$

FOIL

$$= \cancel{\|f\|^2} + 2\|f\|\|g\| + \cancel{\|g\|^2} - (\cancel{\langle f,f\rangle} + \langle f,g\rangle + \langle g,f\rangle + \cancel{\langle g,g\rangle})$$

$$= 2\|f\|\|g\| - (\langle f,g\rangle + \overline{\langle f,g\rangle})$$

$$z+\bar{z} = 2\,\mathrm{Re}\, z$$

$$= 2(\|f\|\|g\| - \mathrm{Re}\langle f,g\rangle)$$

$$\geq 2(\|f\|\|g\| - |\langle f,g\rangle|) \quad (*)$$

Because when we replace Re<f,g> with |<f,g>|, we get a larger quantity that is being subtracted, making the expresssion as a whole smaller (or at least not bigger).

But $(*) \geq 0$ by C-S!

Normed spaces

Definition

norms are abstract versions of lengths

V a fn space. A **norm** on V is $\|\cdot\| : V \to \mathbb{R}$ s.t.:

1. (Positive definite) For all $f \in V$, $\|f\| \geq 0$, and if $\|f\| = 0$, then $f = 0$.
2. (Absolute homogeneity) For all $f \in V$ and $a \in \mathbb{C}$, $\|af\| = |a| \, \|f\|$.
3. (Triangle inequality) For all $f, g \in V$, $\|f + g\| \leq \|f\| + \|g\|$.

A **normed space** is a fn space with a choice of norm.

Example

V is an IP norm, the IP (or L^2) norm on V is a norm as defined above:

- Pos def by defn of IP ✓
- Just proved triangle inequality ✓
- Abs homogeneity: ✓

Other norms

(All norms applied to space of continuous functions on S^1.)

Recall: L^∞ $d(f,g) = \sup\{|f(x)-g(x)| \mid x \in S^1\}$

Example

Consider the L^∞ metric on $V = C^0(S^1)$. If we define

$$\|f\| = d(f,0) = \sup\{|f(x)| \mid x \in \cancel{X}\ S^1\},$$

max diff bet f & 0.

then $\|\cdot\|$ is a norm on V, called the L^∞ **norm** on V.

Example

$\int_{S^1} |f(x)|\, dx$

Let $V = C^0(S^1)$, and define

$$\|f\| = \int_X^{1}_{0} |f(x)|\, dx. = \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|\, dx$$

We call this the L^1 **norm** on V.

Positive defn: The only way to get a nonneg continuous function to have integral = 0 is if function = 0 (PS03).

So we now have 3 different ways to measure the size of a continuous function on S^1:

$$\|f\|_2 = \sqrt{\langle f, f\rangle} = \left(\int_{S^1} |f(x)|^2 \, dx\right)^{1/2}$$

mean squared error

$$\|f\|_1 = \int_{S^1} |f(x)| \, dx$$

mean abs error

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in S^1\}$$

worst case error

The norm metric and limits in normed spaces

Let V be a normed space.

Definition

We define the **norm metric** on V by $d(f, g) = \|f - g\|$.

For a sequence f_n in a normed space V and $f \in V$, to say that $\lim_{n\to\infty} f_n = f$ means that:

$\forall \epsilon > 0$

$\exists N(\epsilon)$ s.t.

If $n > N(\epsilon)$

then $\|f_n - f\| < \epsilon$

Think of f_n as a sequence of approximations to f. This definition of f_n -> f means that the size of the difference between f_n and f goes to 0, where size is measured in the chosen meaning of norm.

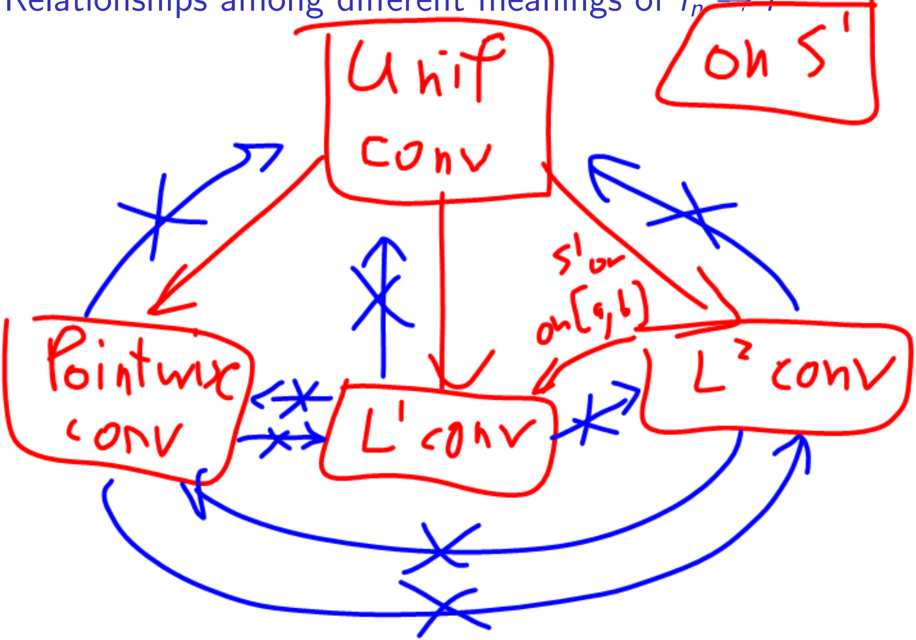
Different meanings of $f_n \to f$

Let $V = C^0([0,1])$, and consider f_n in V. Note that we have now defined $\lim_{n\to\infty} f_n = f$ in four different ways:

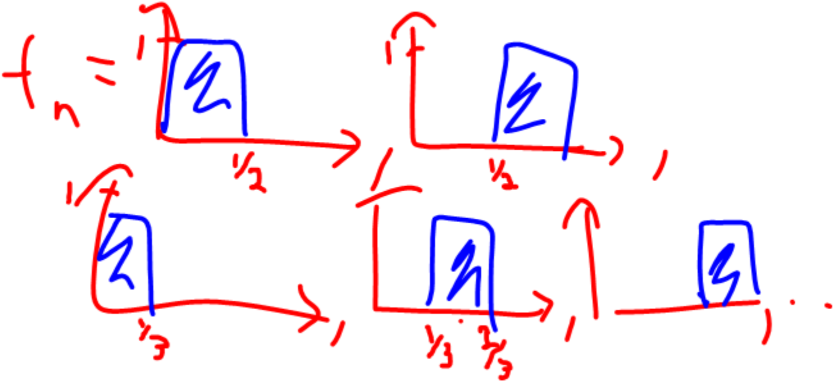
- *Pointwise convergence:* For every $x \in [0,1]$, $\lim_{n\to\infty} f_n(x) = f(x)$.
- *Uniform, or L^∞ convergence:* If $\|\cdot\|_\infty$ is the L^∞ norm on $C^0([0,1])$, then $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$, i.e., f_n converges uniformly to f on $[0,1]$.
- *L^1 convergence:* $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)| \, dx = 0$.
- *L^2 convergence/inner product norm:* $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|^2 \, dx = 0$.

IP norm uses geom of $\langle \, , \rangle$!

Relationships among different meanings of $f_n \to f$



L^1, not pointwise $f=0$



So here, for any point x in [0,1], no matter how large we choose N, there will be some n>N such that f_n(x)=1. So f_n converges to f nowhere pointwise. But integrals converge to 0.

Limit laws in a normed space

Limit laws work in normed spaces pretty much as they work in $\mathbb{C}$.

Theorem

If f_n is a convergent sequence in V, then f_n is bounded.

Theorem

Let f_n and g_n be sequences in V, and suppose that $\lim_{n\to\infty} f_n = f$, $\lim_{n\to\infty} g_n = g$, and $c \in \mathbb{C}$. Then we have that:

1. $\lim_{n\to\infty} cf_n = cf$; *and*
2. $\lim_{n\to\infty} (f_n + g_n) = f + g$.

Proofs are the same too.