Math 131B, Wed Oct 07

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 6.4, 7.1. Reading for Mon: 7.2.
- PS05 due tonight; outline for PS06 due Fri.
- Problem session Fri Oct 09, 10:00–noon on Zoom.

Fourier series

$f_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e_n(x)$

Definition

$f : S^1 \to \mathbb{C}$ integrable, and recall

nth Fourier coeff

$$\hat{f}(n) = \int_0^1 f(x)\,\overline{e_n(x)}\,dx.$$

We define the **Fourier series of f** to be:

$$f(x) \sim \lim_{N\to\infty} f_N(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e_n(x).$$

Note that $\sim$ has no implications about convergence, pointwise or otherwise.

The only trig series that converges uniformly to f

If a trig series converges uniformly to f, it must be the Fourier series of f:

Theorem

Let $f : S^1 \to \mathbb{C}$ be integrable and let

$$g_N(x) = \sum_{n=-N}^{N} c_n e_n(x)$$

be a sequence of trigonometric polynomials such that g_N converges to f uniformly on $[0, 1]$ (i.e., on S^1). Then

$$c_n = \hat{f}(n) = \int_0^1 f(x)\,\overline{e_n(x)}\,dx.$$

q

(prove for k, not n)

Pf (4) $g_N(x) = \sum_{n=-N}^{N} c_n e_n(x)$, $g_N \to f$ unif.

Then for $k \in \mathbb{Z}$, $\int_0^1 f(x)\overline{e_k(x)}\,dx$

f(x) is the sum of a series, so we analyze that series by writing it as the limit of its partial sums g_N.

$$= \int_0^1 \left(\lim_{N\to\infty} g_N(x)\right)\overline{e_k(x)}\,dx$$

$$= \lim_{N\to\infty} \int_0^1 g_N(x)\overline{e_k(x)}\,dx$$

$\boxed{\text{unif conv}}$

If we have uniform conv, we can pass lim thru integral

$$= \lim_{N\to\infty} \int_0^1 \left(\sum_{n=-N}^{N} c_n e_n(x)\right) \overline{e_k(x)}\,dx$$

Finite sums can always pass thru integrals

$$= \lim_{N\to\infty} \sum_{n=-N}^{N} c_n \underbrace{\int_0^1 e_n(x)\,\overline{e_k(x)}\,dx}_{=0 \text{ unless } n=k,\ \text{get } 1}$$

When N>|k|,
every term in the purple sum = 0, except the term where n=k. So taking the limit as N -> infinity, we can throw away every term in the sum except when n=k.

$$= c_k \cdot 1 = c_k. \text{ So } \boxed{c_k = \int_0^1 f(x)\,\overline{e_k(x)}\,dx.}$$

Let's be less ambitious

Before we can answer:

MAIN Q: *When does* $\sum_{n\in\mathbb{Z}} \hat{f}(n)e_n(x)$ *converge to* $f(x)$*?*

And in what sense?

Let's tackle:

When does $\sum_{n\in\mathbb{Z}} \hat{f}(n)e_n(x)$ *converge?*

The surprising key: f' cont. $\widehat{\frac{df}{dx}}(n)$

Theorem

For $f \in C^1(S^1)$ *and* $n \in \mathbb{Z}$*, we have that*

$$\widehat{f'}(n) = (2\pi i n)\hat{f}(n).$$

Proof: PS06. (Parts!!!!)

NB:

$$\widehat{\frac{df}{dx}} = (2\pi i n)\hat{f}$$

So taking Fourier coefficients turns the analytic operation of d/dx into the algebraic operation of multiplication by (2 pi i n).

$$\widehat{f'}(n) = (2\pi i n)\hat{f}(n)$$

Differentiability implies decay of coefficients

Fourier

A broadly useful principle!

Theorem

For $f : S^1 \to \mathbb{C}$, we have that:

1. *If f is continuous (i.e., $f \in C^0(S^1)$), then there exists some constant $K_0 > 0$, independent of n, such that $\left|\hat{f}(n)\right| \leq K_0$ for all $n \in \mathbb{Z}$.*
2. *For any integer $r \geq 1$, if $f \in C^r(S^1)$, then there exists some constant $K_r > 0$, independent of n, such that $\left|\hat{f}(n)\right| \leq \dfrac{K_r}{|n|^r}$ for all $n \in \mathbb{Z}$, $n \neq 0$.*

E.g., if $f \in C^3$, then as $|n| \to \infty$, $\hat{f}(n) \to 0$ at rate of $1/|n|^3$.

Proof:

1 A $f \in C^0(S^1)$

$$|\hat{f}(n)| = \left| \int_0^1 f(x)\,\overline{e_n(x)}\,dx \right|$$

△ ineq for $\int$

$$\leq \int_0^1 |f(x)\,\overline{e_n(x)}|\,dx$$

$$= \int_0^1 |f(x)|\,\underbrace{|\overline{e_n(x)}|}_{=1}\,dx$$

$$= \int_0^1 |f(x)|\,dx = K_0.$$

② Ind on r; $r=0$

(A) If $g \in C^r$, $r \geq 0$, $|\hat{g}(n)| \leq \frac{K_r}{|n|^r}$

(4) $f \in C^{r+1}$, so $f' \in C^r$.

$$\Rightarrow |\widehat{f'}(n)| \leq \frac{K_r}{|n|^r}$$

the key step

$$\Rightarrow |(2\pi i n)\hat{f}(n)| \leq \frac{K_r}{|n|^r} \qquad n \neq 0$$

$$|\hat{f}(n)| \leq \frac{K_r}{2\pi |n|^{r+1}}$$

Take $K_{r+1} = \frac{K_r}{2\pi}$

Convergence of Fourier series of C^2 functions

Theorem

If $f \in C^2(S^1)$, then the Fourier series of f converges absolutely and uniformly to some continuous function g such that for all $n \in \mathbb{Z}$, $\hat{g}(n) = \hat{f}(n)$.

Proof: PS06.

But it doesn't obviously follow that $f = g$. What if $\hat{g}(n) = \hat{f}(n)$ for all $n \in \mathbb{Z}$, but $f \neq g$?

To prove that $f = g$, we need either lots of hard detailed work or more abstract theory. We go in the abstract theory direction. . . .

<u>Hilbert Spaces</u>

Inner product spaces

like dot prod

Definition

V be a function space. An **inner product** on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that satisfies:

1. (Linear in first variable) For any $f, g, h \in V$ and $a, b \in \mathbb{C}$, we have that $\langle af + bg, h \rangle = a \langle f, h \rangle + b \langle g, h \rangle$.
2. (Hermitian) For any $f, g \in V$, $\langle g, f \rangle = \overline{\langle f, g \rangle}$. Note that consequently, for any $f \in V$, $\langle f, f \rangle = \overline{\langle f, f \rangle}$ must be in $\mathbb{R}$.
3. (Positive definite) For any $f \in V$, $\langle f, f \rangle \geq 0$, and if $\langle f, f \rangle = 0$, then $f = 0$.

An **IP space** is a V along with a particular choice of inner product.

Definition

V an IP space. For $f \in V$, we define the **norm** of f to be $\|f\| = \sqrt{\langle f, f \rangle}$. We call $\|f\| = \sqrt{\langle f, f \rangle}$ the **inner product norm**, or L^2 **norm**, on V.

length

$\langle f, f \rangle = \|f\|^2$

Note If $\langle\,,\rangle$ an IP on V:

$$\langle f, ag+bh\rangle = \bar{a}\langle f,g\rangle + \bar{b}\langle f,h\rangle$$

(skew-linear)

Note $\|f\| = \sqrt{\langle f,f\rangle}$ $\sqrt{\ }$ bad

$\|f\|^2 = \langle f,f\rangle$ algebra good

Examples

Example

For $V = \mathbb{C}^n$, the **dot product**

stew

$$\langle (v_1, \ldots, v_n), (w_1, \ldots, w_n) \rangle = v_1\overline{w}_1 + \cdots + v_n\overline{w}_n$$

is an IP on V.

$\int_0^1$ or $\int_a^b$

Example

Let $X = [a, b]$ or S^1, and let $V = C^0(X)$. Then for $f, g \in V$,

$$\langle f, g \rangle = \int_X f(x)\overline{g(x)}\, dx$$

is an IP on V (PS06), which we call the L^2 **inner product**. Note that

$$\hat{f}(n) = \int_0^1 f(x)\overline{e_n(x)}\,dx = \langle f, e_n \rangle.$$

Orthogonality

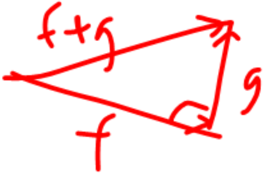
Definition

Let V be an inner product space. For $f, g \in V$, to say that f is **orthogonal** to g means that $\langle f, g \rangle = 0$.

Theorem (Pythagorean Theorem)

Let V be an inner product space. If $f, g \in V$ are orthogonal, then $\|f+g\|^2 = \|f\|^2 + \|g\|^2$.

Proof:



$\|f+g\|^2$

$= \langle f+g, f+g \rangle$ bilinearity (linear in 1, skew in 2)

$= \langle f,f \rangle + \langle g,f \rangle + \langle f,g \rangle + \langle g,g \rangle$

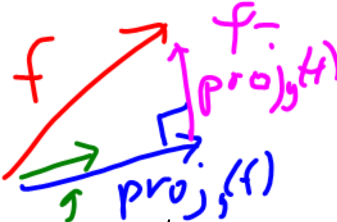
$= \|f\|^2 + 0 + 0 + \|g\|^2$ orth ☺

Projection

Definition

Let V be an inner product space, and $g \neq 0$ in V. For $f \in V$, we define the **projection of** f **onto** $g$to be

$$\mathrm{proj}_g(f) = \frac{\langle f, g\rangle}{\langle g, g\rangle} g.$$



Theorem

Let V be an inner product space, and let g be a nonzero element of V. For $f \in V$, we have:

$$\begin{aligned}
\left\langle \mathrm{proj}_g(f), g \right\rangle &= \langle f, g\rangle\,, \\
\left\langle f - \mathrm{proj}_g(f), g \right\rangle &= 0, \\
\left\langle f - \mathrm{proj}_g(f), \mathrm{proj}_g(f) \right\rangle &= 0, \\
\left\| \mathrm{proj}_g(f) \right\| &\leq \| f \|\,.
\end{aligned}$$

Cauchy-Schwarz and triangle

Theorem

V an IP space. For $f, g \in V$, we have:

1. *(Cauchy-Schwarz inequality)* $|\langle f, g\rangle| \leq \|f\| \, \|g\|$; *and*
2. *(Triangle inequality)* $\|f + g\| \leq \|f\| + \|g\|$.

Proof of C-S: First show: $|\langle f, g\rangle| = \left\|\mathrm{proj}_g(f)\right\| \, \|g\|$.

Proof of triangle inequality

Normed spaces

Definition

V a fn space. A **norm** on V is $\|\cdot\| : V \to \mathbb{R}$ s.t.:

1. (Positive definite) For all $f \in V$, $\|f\| \geq 0$, and if $\|f\| = 0$, then $f = 0$.
2. (Absolute homogeneity) For all $f \in V$ and $a \in \mathbb{C}$, $\|af\| = |a| \, \|f\|$.
3. (Triangle inequality) For all $f, g \in V$, $\|f + g\| \leq \|f\| + \|g\|$.

A **normed space** is a fn space with a choice of norm.

Example

V is an IP norm, the IP norm on V is a norm as defined above:

- Pos def by defn of IP
- Just proved triangle inequality
- Abs homogeneity: