Counterexamples in pointwise convergence: The six NO's

QB: If lim f_n = f, f_n all bounded, must f be bounded?
NO: f(x)=1/1-x for -1<x<1, f_n = 1 + x + ... + x^n:

```
> geomsums[0] := 1;
```

$$geomsums_0 := 1 \qquad (1)$$

```
> for n from 1 to 5 do geomsums[n] := geomsums[n-1] + x^n end do;
```

$$geomsums_1 := 1 + x$$

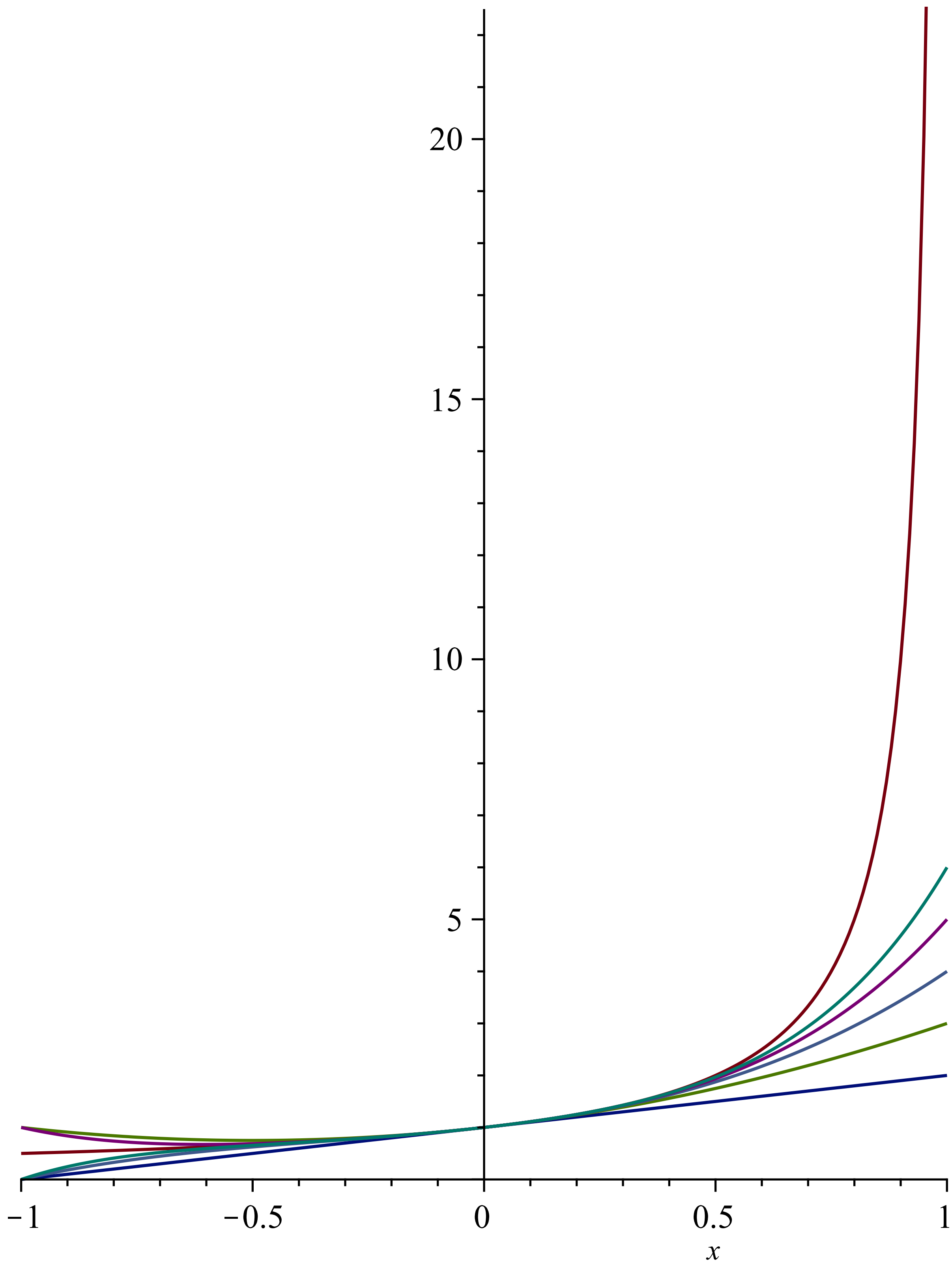
$$geomsums_2 := x^2 + x + 1$$

$$geomsums_3 := x^3 + x^2 + x + 1$$

$$geomsums_4 := x^4 + x^3 + x^2 + x + 1$$

$$geomsums_5 := x^5 + x^4 + x^3 + x^2 + x + 1 \qquad (2)$$

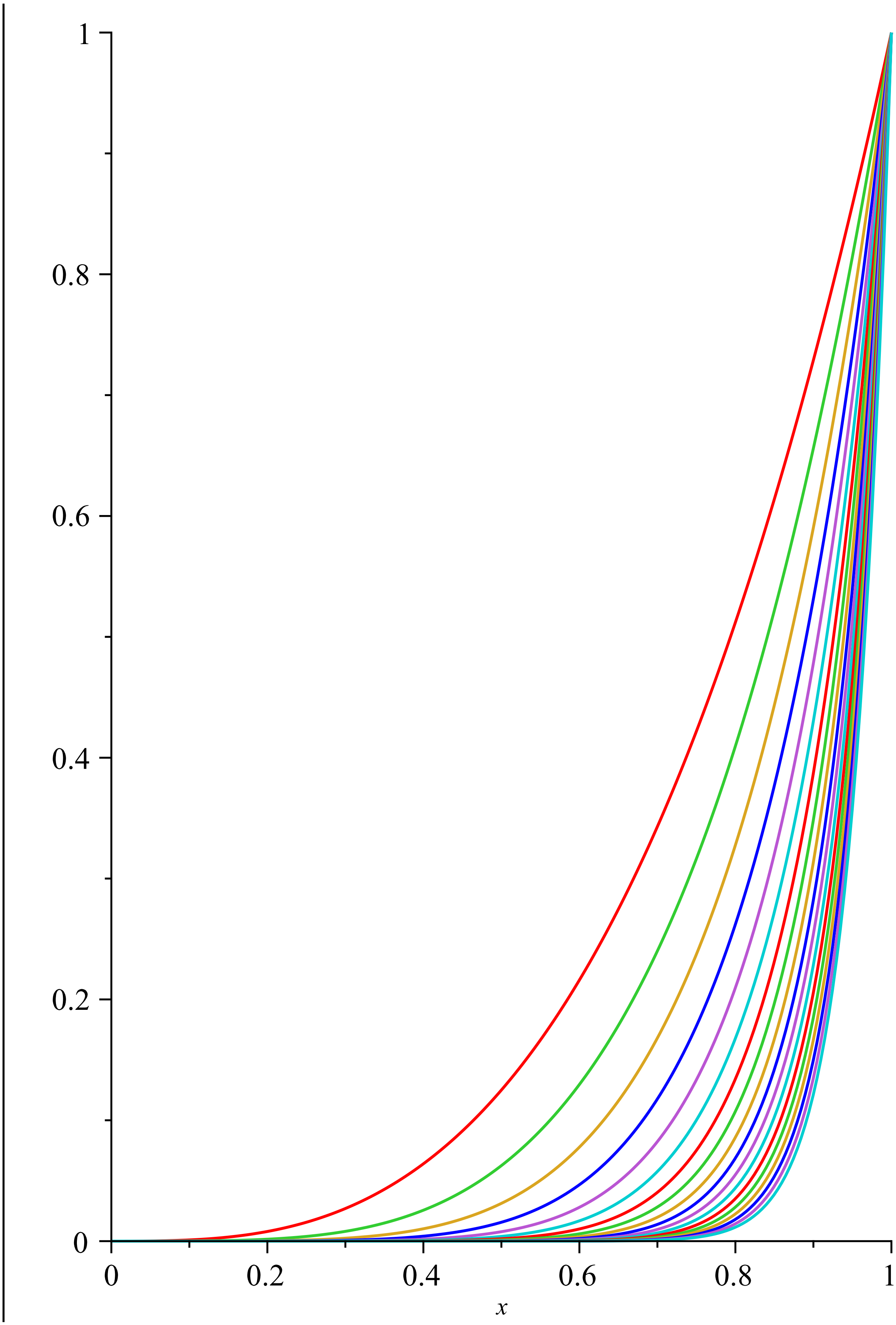
```
> plot([1/(1-x), seq(geomsums[n], n=1..5)], x = -1..1);
```



All domains = [0,1] from here onwards.

QC: If lim f_n = f, f_n all continuous, must it be the case that f is continuous?
QD1: If lim f_n = f, f_n all differentiable, must it be the case that f is differentiable?
NO: f(x) = 0 if 0<=x<1, f(1)=1, and f_n = x^n:

```
> plot([seq(x^n,n=3..20)],x=0..1);
```



1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
x

Note: f_n converges to f, but not uniformly.

QD2: If lim f_n = f, f_n and f all differentiable, must it be the case that lim f '_n = f '?
NO: Take g(x)=0 and:

```
> g_n := x^(n+1)/(n+1);
```

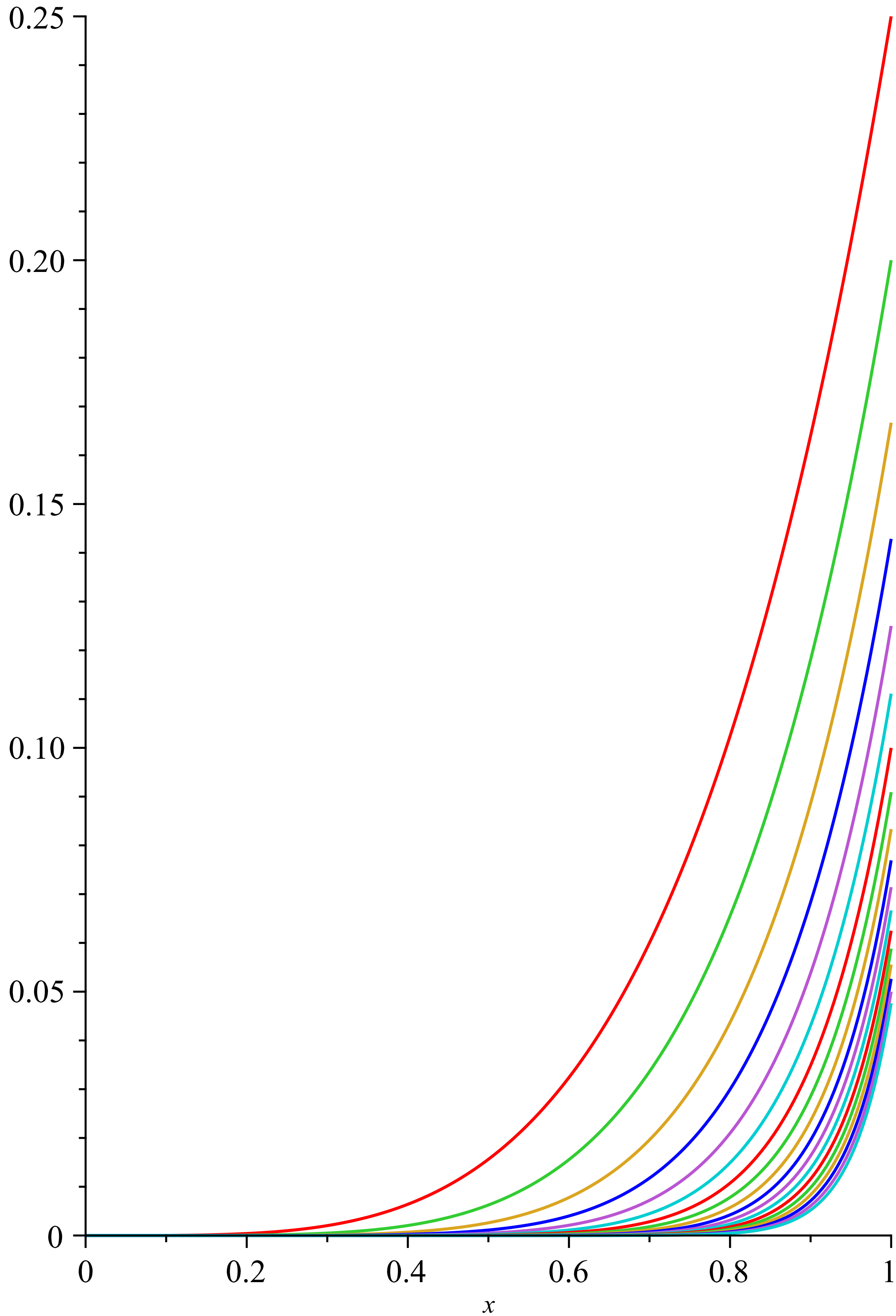
$$g\_n := \frac{x^{n+1}}{n+1} \tag{3}$$

```
> simplify(diff(g_n,x),assume=positive);
```

$$x^n \tag{4}$$

When x=1, lim(x^n)=1. But lim (g_n(x)) =0 for all x in [0,1], and in fact, convergence of g_n is uniform on [0,1]. (Slowest convergence is at x=1, but even there, g_n converges like 1/(n+1).)

```
> plot([seq(x^(n+1)/(n+1),n=3..20)],x=0..1);
```



QI1: If lim f_n = f, f_n all integrable, must it be the case that f is integrable?

NO: Example is k(x) = 1 if x is rational, 0 if x irrational; k_n(x) = 1 if x=p/q (least terms) and q<=n, 0 otherwise. lim k_n = k, but "finitely many points at a time", so each g_n is continuous except at finitely many points, and therefore integrable.

QI2: If lim f_n = f, f_n and f all integrable, must it be the case that lim of:

```
> int(f_n(x),x=0..1);
```

$$\int_0^1 f\_n(x)\,dx \tag{5}$$

is equal to:

```
> int(f(x),x=0..1);
```

$$\int_0^1 f(x)\,dx \tag{6}$$

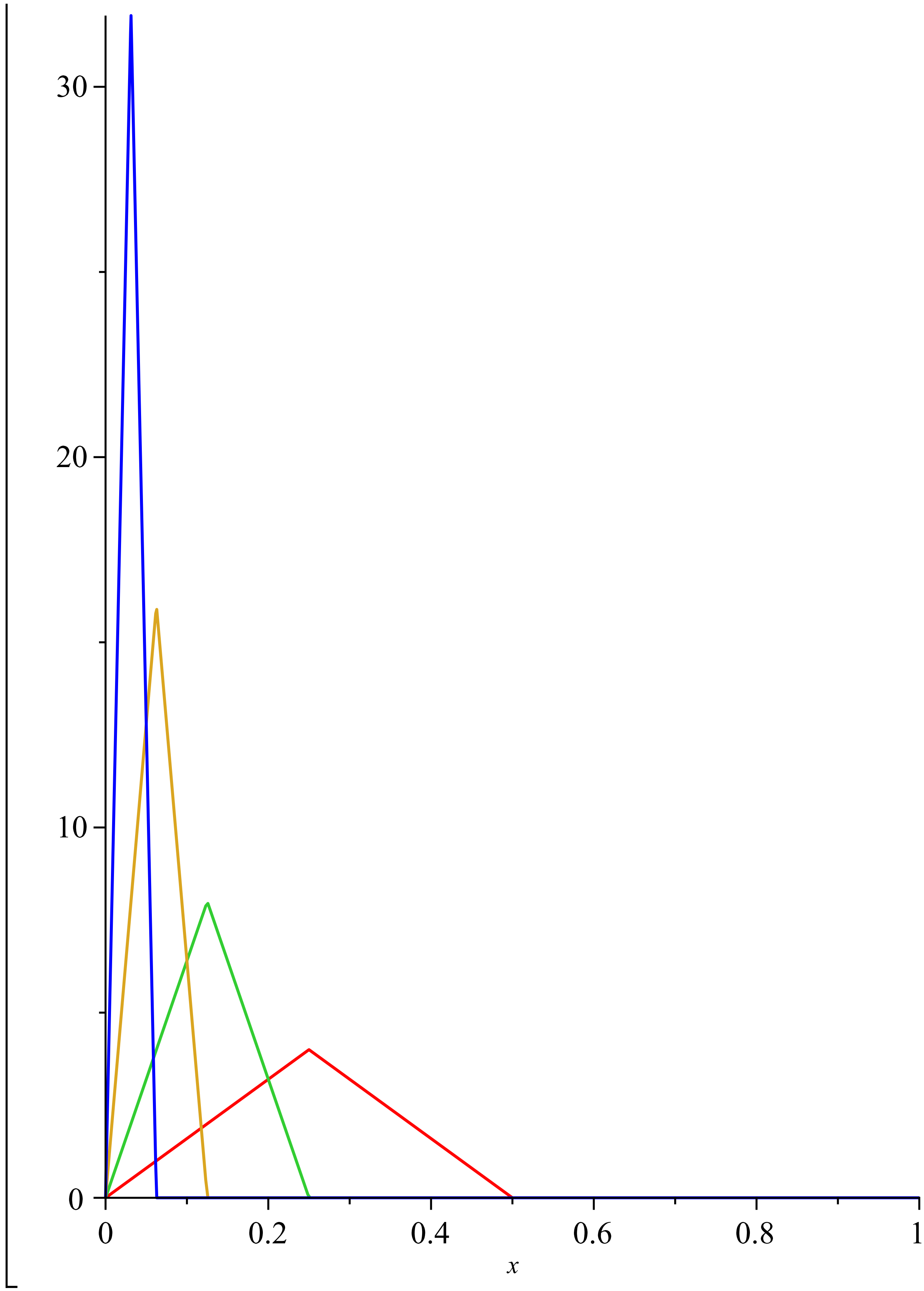
NO, not even if all functions continuous: h(x)=0 and h_n given by:

```
> h_n = piecewise(x<1/2^(n+1),2^(2*n+2)*x,
  x<1/2^n,2^(2*n+2)*((1/2^n)-x),0);
```

$$h\_n = \begin{cases} 2^{2n+2}x & x < \frac{1}{2^{n+1}} \\ 2^{2n+2}\left(\frac{1}{2^n} - x\right) & x < \frac{1}{2^n} \\ 0 & \textit{otherwise} \end{cases} \tag{7}$$

```
> plot([seq(piecewise(x<1/2^(n+1),2^(2*n+2)*x,
  x<1/2^n,2^(2*n+2)*((1/2^n)-x),0),n=1..4)],x=0..1);
```

(Each integral of h_n = 1 because each triangle is half as wide and twice as high as the previous one.)


30
20
10
0
0
0.2
0.4
0.6
0.8
1
x