Math 131B, Mon Sep 14

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 3.5. Reading for Wed: 4.1–4.2.
- Outline for PS04 (not written yet) due Wed Sep 23.
- Next problem session Fri Sep 18, 10:00–noon on Zoom: Exam review
- Zoom proctoring rehearsal **TODAY**.
- **Exam 1 moved to Mon Sep 21**, to cover 2.1–2.5, 3.1–3.4, i.e., PS01–03.

Exam rehearsal at 11:35am

2 pieces paper

1. Please have a clear workspace ready where you can write.
2. Please have some kind of camera ready. First position the camera so I can see your face, and later so I can see your workspace.
3. Please have the Gradescope assignment page "Exam rehearsal" open and ready to go.

Questions?

Exam format

Types of questions:

- Computations
- Proofs
- True/false with justification

For last type, if true, write "True" for full credit; if false, e.g.:

(True/False) For $a, b \in \mathbb{R}$, if $a \geq b$, then $-a \geq -b$.

Need to write "False" and give justification.

False Take $a=7$, $b=5$. then $7 \geq 5$ but $-7 \not\geq -5$.

False If $a > b$, then $-a < -b$ by order axioms.

Last time

- Algebraic properties of definite integrals.
- Classes of functions that are integrable: Continuous functions, piecewise continuous functions, monotone functions.

  bd

- Triangle inequality for integrals:

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Questions?

Fundamental Theorems of Calculus

in $\mathbb{R}$

Ex e^{ix}

Theorem (FTC I)

Let I be an interval, $a \in I$, let $f : I \to \mathbb{C}$ be integrable on any closed subinterval of I, and let

bd

$$F(x) = \int_a^x f(t)\,dt.$$

Indefinite integral of f (you know, the +C thing)

Then F is (uniformly) continuous on I, and furthermore, if f is continuous at some $b \in I$, then F is differentiable at b and $F'(b) = f(b)$.

I.e., if f cont, then derivative of indef integral of f gives back f.

PS 04

Theorem (FTC II)

Let $F : [a, b] \to \mathbb{C}$ be continuously differentiable. Then

$$F(b) - F(a) = \int_a^b \frac{dF}{dx}\,dx.$$

Total change in F from a to b = Definite integral of rate of change

Consequences of FTC 2

Theorem (Substitution) Chain rule + FTC 2 = substitution

$X \subseteq \mathbb{C}$; $u : [a,b] \to X$, $f : X \to \mathbb{C}$ cont diff'ble. Then

$$\int_a^b f'(u(x))u'(x)\,dx = f(u(b)) - f(u(a)).$$

In particular, if X a subinterval of $\mathbb{R}$ and $g : X \to \mathbb{C}$ cont,

$$\int_a^b g(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} g(u)\,du.$$

u-sub

$\int u\,dv = uv - \int v\,du$

Theorem (Integration by parts)

Let $f, g : [a,b] \to \mathbb{C}$ cont diff'ble. Then Prod rule + FTC 2 = parts

$$\int_a^b f(x)g'(x)\,dx = \underbrace{f(b)g(b) - f(a)g(a)}_{} - \int_a^b g(x)f'(x)\,dx.$$

$$f(x)g(x)\Big]_a^b$$

YOU WOULD NOT BELIEVE HOW IMPORTANT INTEGRATION BY PARTS IS FOR UPPER-LEVEL (GRADUATE) ANALYSIS!!!!

$$F(x) = \int_a^x f(t)\,dt$$

(A) f $\int$-ble on $[b,c] \subseteq I$

Proof of FTC I: $F(x)$ continuous

dstr of $\int$ble

Suppose $|f(t)| \leq M$ (integrable implies bounded). Then

$$|F(x) - F(b)| = \left| \int_a^x f(t)\,dt - \int_a^b f(t)\,dt \right|$$

$$= \left| \int_a^x f(t)\,dt + \int_b^a f(t)\,dt \right|$$

additivity of domain

~~So given $\epsilon > 0$, let $\delta(\epsilon) =$~~ .

$$= \left| \int_b^x f(t)\,dt \right|$$

~~Then if $|x - b| < \delta(\epsilon)$~~, we have:

We would like |F(x)-F(b)| to be small when |x-b| is small.
Use triangle inequality for integrals:

$$\leq \left| \int_b^x |f(t)|\,dt \right| \leq \left| \int_b^x M\,dt \right| = M|x-b|$$

So $|F(x)-F(b)| \leq M|x-b|$.

WTS F cont at b. ϵ-δ:

Ⓐ $\epsilon > 0$

Let $\delta(\epsilon) = \epsilon/M$

Ⓐ $|x-b| < \delta(\epsilon) = \frac{\epsilon}{M}$

Then

$|F(x)-F(b)| \leq M|x-b| < M\frac{\epsilon}{M} = \epsilon$

Ⓒ $|F(x)-F(b)| < \epsilon$

Outline for ϵ-δ cont.

Proof of FTC I: If f cont at b, $F'(b) = f(b)$

$F(x) = \int_a^x f(t)\,dt$

Suppose f cont at b. For $x \neq b$, observe:

$$\left| \frac{F(x) - F(b)}{x - b} - f(b) \right| =$$

Since f cont at b, there exists $\delta_0(\epsilon_0) > 0$ such that if $|x - b| < \delta_0(\epsilon_0)$, then $|f(x) - f(b)| < \epsilon_0$.

Defn of F diff'ble at b and F'(b) = f(b):

$$\lim_{x \to b} \frac{F(x) - F(b)}{x - b} = f(b)$$

Assume $\epsilon > 0$. Let $\delta(\epsilon) =$.

(A) $|x - b| < \delta(\epsilon), x \neq b$

(C) $\left| \frac{F(x) - F(b)}{x - b} - f'(b) \right| < \epsilon$

Ch. 4: Infinite series in $\mathbb{C}$

Defn: a_n seq in $\mathbb{C}$. Series $\displaystyle\sum_{n=k}^{\infty} a_n$ defined as:

- Define sequence of partial sums s_N by setting $s_k = a_k$ and, for $N \geq k$, setting $s_{N+1} = s_N + a_{N+1}$. In other words:

$$\begin{aligned}
s_k &= a_k \\
s_{k+1} &= s_k + a_{k+1} = a_k + a_{k+1} \\
s_{k+2} &= s_{k+1} + a_{k+2} = a_k + a_{k+1} + a_{k+2} \\
s_{k+3} &= s_{k+2} + a_{k+3} = a_k + a_{k+1} + a_{k+2} + a_{k+3} \\
&\vdots \\
s_{N+1} &= s_N + a_{N+1} = a_k + a_{k+1} + \cdots + a_{N+1} \\
&\vdots
\end{aligned}$$

Ch. 4: Infinite series in $\mathbb{C}$

- To say that $\sum_{n=k}^{\infty} a_n$ converges means that the sequence of partial sums s_N converges, in which case we define

$$\sum_{n=k}^{\infty} a_n = \lim_{N\to\infty} s_N.$$

Cauchy criterion for series and comparison

Cauchy completeness implies:

Corollary (Cauchy Criterion for Series)

The series $\sum a_n$ *converges if and only if for every* $\epsilon > 0$, *there exists some* $N(\epsilon)$ *such that if* $m, k \in \mathbb{Z}$ *and* $m, k > N(\epsilon)$, *then*

$$\left| \sum_{n=k}^{m} a_n \right| < \epsilon.$$

Corollary (Comparison Test)

a_n, b_n *sequences,* $b_n \geq 0$.

1. *If* $\sum b_n$ *converges and* $|a_n| \leq b_n$ *for all* n *(or sufficiently large* n*), then* $\sum a_n$ *converges.*
2. *If* $\sum b_n$ *diverges,* $a_n \geq 0$, *and* $b_n \leq a_n$ *for all* n *(or sufficently large* n*), then* $\sum a_n$ *diverges.*

Absolute convergence

Corollary

If $\sum |a_n|$ *converges, then so does* $\sum a_n$. $\square$

Definition

To say that $\sum a_n$ **converges absolutely** means that $\sum |a_n|$ converges (and therefore, so does $\sum a_n$).

Two-sided series

Definition

Let $a_n : \mathbb{Z} \to \mathbb{C}$ be a **two-sided sequence**, which we can think of as a sequence a_n where n goes to both $+\infty$ and $-\infty$. To say that the the corresponding **two-sided series** $\sum_{n \in \mathbb{Z}} a_n$ **converges** means that for some $k \in \mathbb{Z}$, both $\sum_{n=k-1}^{-\infty} a_n$ and $\sum_{n=k}^{\infty} a_n$ converge, in which case we define

$$\sum_{n \in \mathbb{Z}} a_n = \sum_{n=k-1}^{-\infty} a_n + \sum_{n=k}^{\infty} a_n.$$

Important because in our approach, Fourier series are two-sided series.

Synchronous convergence

Defn: Given two-sided sequence a_n, synchronous partial sums s_N are

$$\begin{aligned}
s_0 &= a_0 \\
s_1 &= s_0 + a_{-1} + a_1 = a_{-1} + a_0 + a_1 \\
s_2 &= s_1 + a_{-2} + a_2 = a_{-2} + a_{-1} + a_0 + a_1 + a_2 \\
&\vdots \\
s_N &= s_{N-1} + a_{-N} + a_N = \sum_{n=-N}^{N} a_n \\
&\vdots
\end{aligned}$$

Synchronous convergence, cont.

To say $\sum_{n\in\mathbb{Z}} a_n$ converges synchronously means the sequence of synchronous partial sums s_N does. Furthermore, if $\sum_{n\in\mathbb{Z}} a_n$ converges synchronously, we define the **synchronous sum** of $\sum_{n\in\mathbb{Z}} a_n$ to be

$$\sum_{n\in\mathbb{Z}} a_n = \lim_{N\to\infty} s_N = \lim_{N\to\infty} \sum_{n=-N}^{N} a_n.$$