Math 131B, Wed Sep 09

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 3.4. Reading for Mon: 3.5.
- Outline for PS03 due 11pm, complete PS03 due Mon Sep 14.
- Next problem session Fri Sep 11, 10:00–noon on Zoom.
- Zoom proctoring rehearsal Mon Sep 14. Details over the weekend, but have blank paper ready and be ready to turn on your camera on Mon.

  +ch 1
- **Exam 1 moved to Mon Sep 21**, to cover 2.1–2.5, 3.1–3.4.

Last time

Defn of $\int_a^b v(x)\,dx$ and:

Lemma (Sequential Criteria for Integrability)

Let $v : [a,b] \to \mathbb{R}$ be bounded. Then the following are equivalent.

1. *v is integrable on $[a,b]$.*
2. *There exists a sequence of partitions P_n such that $\lim_{n\to\infty}(U(v;P_n) - L(v;P_n)) = 0$.*
3. *For any $\epsilon > 0$, there exists a partition P such that $U(v;P) - L(v;P) < \epsilon$.*

Not by defn, but every integral is a limit of Riemann sums!

Furthermore, if condition (2) holds, then

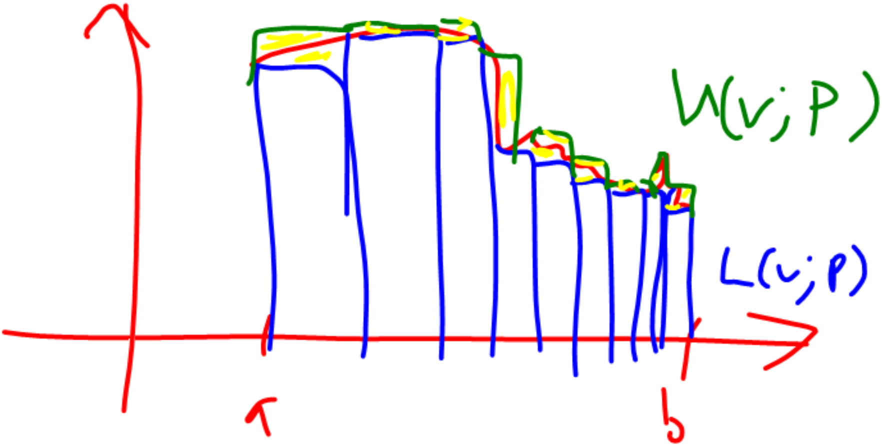
$$\lim_{n\to\infty} L(v;P_n) = \int_a^b v(x)\,dx = \lim_{n\to\infty} U(v;P_n).$$

So you can prove stuff about integral using limits.

Picture of sequential criterion, ϵ version (3)

If, given epsilon, we can choose P such that: total  $< \epsilon$

Then v integrable.



I.e.: To prove v integrable, need to make yellow area small.

Integrals of complex functions $\boxed{3.3}$

$f(x) = u(x) + i v(x)$

Definition

Let $f : [a, b] \to \mathbb{C}$ be bounded, and let u and v be the real and imaginary parts of f. To say that f is integrable means that both u and v are integrable, in which case we define

$$\int_a^b f(x)\, dx = \int_a^b u(x)\, dx + i \int_a^b v(x)\, dx.$$

I.e., define integral of a complex function via integral of its real and imaginary parts — so most facts about integrals of complex functions follow by applying real facts twice.

Sequential criterion for complex integrability

(Complex-valued version of (1) <=> (3) from seq crit for real-valued functions)

Lemma

Let $f : [a, b] \to \mathbb{C}$ be bounded, and for any partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ and $1 \le i \le n$, define

wiggle = max possible difference between two output from subinterval i

$$\mu(f; P, i) = \sup\{|f(x) - f(y)| \mid x, y \in [x_{i-1}, x_i]\},$$

$$E(f; P) = \sum_{i=1}^{n} \mu(f; P, i)(\Delta x)_i.$$

Yellow area from previous picture now replaced by "weighted max wiggle"

Then the following are equivalent.

1. *f is integrable on $[a, b]$.*
2. *For any $\epsilon > 0$, there exists a partition P such that $E(f; P) < \epsilon$.*

Algebraic facts about the integral 3.4

Theorem

Let $v, w : [a, b] \to \mathbb{R}$ integrable, $c > 0$. Then $v + w$, cv, and $-v$ are integrable on $[a, b]$, and

Integration is linear:

$$\int_a^b (v(x) + w(x))\, dx = \int_a^b v(x)\, dx + \int_a^b w(x)\, dx,$$

$$\int_a^b cv(x)\, dx = c \int_a^b v(x)\, dx,$$

$$\int_a^b (-v(x))\, dx = - \int_a^b v(x)\, dx.$$

$$\int_a^b (cv + dw)\, dx = c \int_a^b v\, dx + d \int_a^b w\, dx$$

Furthermore, if $v(x) \leq w(x)$ for all $x \in [a, b]$,

$$\int_a^b v(x)\, dx \leq \int_a^b w(x)\, dx.$$

(1) <=> (2)

Why: Rewrite integrals as limits of Riemann sums.

Additivity of domain

PS03

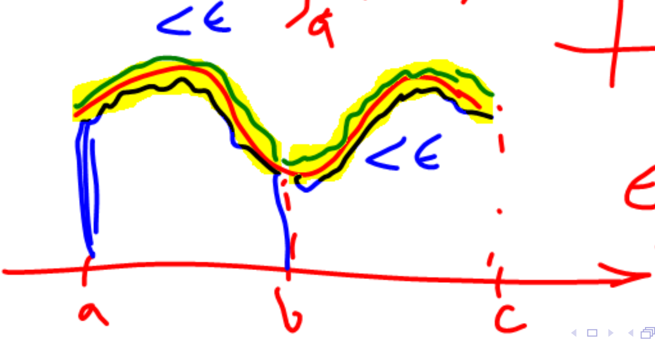
Theorem

For $a < b < c$, let $f : [a, c] \to \mathbb{C}$ be integrable on $[a, b]$ and $[b, c]$. Then f is integrable on $[a, c]$ and

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Picture:

$\int_a^c v(x)\,dx$



ε version of seq crit

Example: Monotone functions are integrable

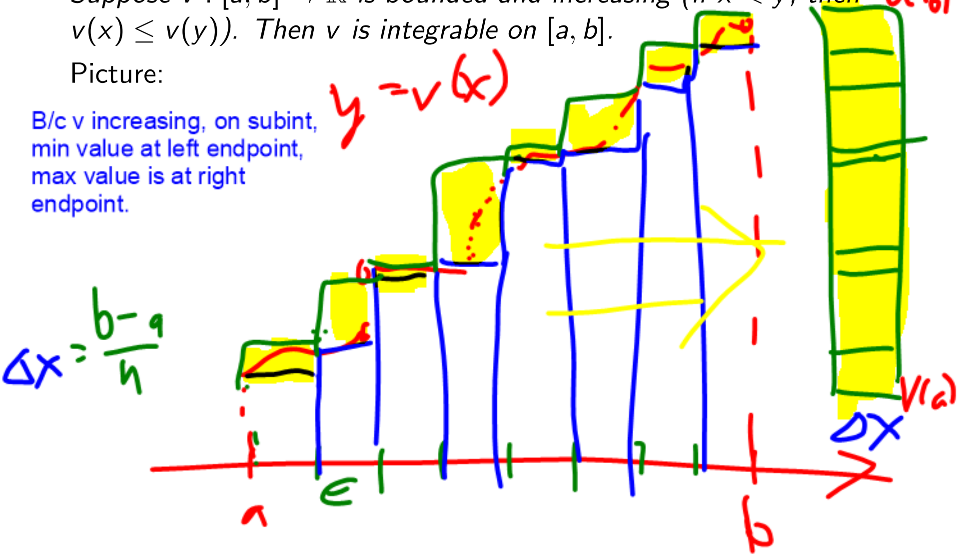
Moral of this result: Non-integrable functions have infinite wiggle

Theorem

Suppose $v : [a, b] \to \mathbb{R}$ is bounded and increasing (if $x < y$, then $v(x) \leq v(y)$). Then v is integrable on $[a, b]$.

Picture:

B/c v increasing, on subint, min value at left endpoint, max value is at right endpoint.



Proof:

Want $(r(b)-r(a))\Delta x < \epsilon$

$(r(b)-r(a))\frac{b-a}{n} < \epsilon$

$n > \frac{(r(b)-r(a))(b-a)}{\epsilon}$

$\forall\ \epsilon > 0$

Choose $n \in \mathbb{Z}$

s.t. $n > \frac{(r(b)-r(a))(b-a)}{\epsilon}$

so if $\Delta x = \frac{b-a}{n}$, then $(r(b)-r(a))\Delta x < \epsilon$.

Let $P = \{x_0 = a, x_1, \ldots, x_n = b\}$

s.t. $x_i = x_{i-1} + \Delta x$ (i.e, n equal pieces)

$$U(v;P) = \sum_{i=1}^{n} M(v;P,i)\,\Delta x$$

$$= \sum_{i=1}^{n} v(x_i)\,\Delta x$$

b/c $v \nearrow$, max at RH endpt of subint i

$$L(v;P) = \sum_{i=1}^{n} m(v;P,i)\,\Delta x$$

$$= \sum_{i=1}^{n} v(x_{i-1})\,\Delta x$$

x_{i-1} x_i

$j = i-1$

$$= \sum_{j=0}^{n-1} v(x_j)\,\Delta x$$

So

$$U(v;P) - L(v;P)$$

$$= \underbrace{\sum_{i=1}^{n} v(x_i)\Delta x - \sum_{i=0}^{n-1} v(x_i)\Delta x}_{\text{mid term cancel}}$$

$$= v(x_n)\Delta x - v(x_0)\Delta x$$

$$= (v(a) - v(b))\Delta x < \epsilon$$

Continuity implies integrability

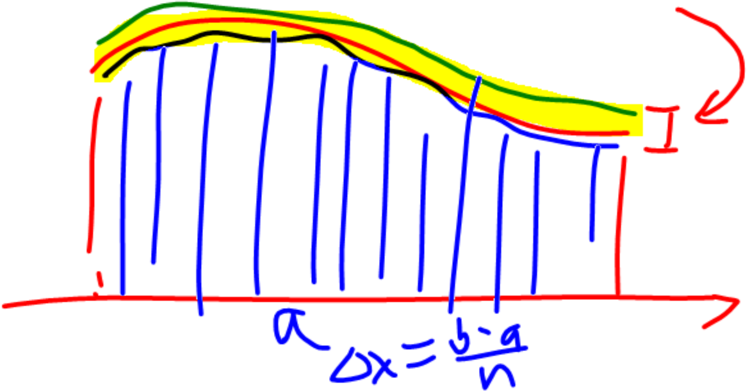
PS03

Theorem

If $f : [a, b] \to \mathbb{C}$ continuous, then f integrable on $[a, b]$.

Picture:

Idea: Use uniform continuity to make sure that difference between max and min on any subinterval is small.



New integrable functions from old

$\boxed{\text{abs value}}$

product

Theorem

If $f, g : [a,b] \to \mathbb{C}$ are integrable, then $|f(x)|$, $f(x)^2$, and $f(x)g(x)$ are also integrable on $[a,b]$. If we also assume that f and g are real-valued, then $\min(f(x), g(x))$ and $\max(f(x), g(x))$ are integrable on $[a,b]$.

min, max

Why: Follows from more general lemma:

Lemma

If $f : [a,b] \to \mathbb{C}$ is integrable, and $\varphi : \mathbb{C} \to \mathbb{C}$ is continuous, then $\varphi \circ f : [a,b] \to \mathbb{C}$ is integrable.

(Proof of more general lemma is complicated!)

Triangle inequality for integrals

Theorem

If $f : [a, b] \to \mathbb{C}$ integrable,

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Why is this like the triangle inequality?

$\triangle$ ineq $\qquad |z+w| \leq |z| + |w|$

for sums $\qquad \left| \sum_{i=1}^{n} z_n \right| \leq \sum_{i=1}^{n} |z_n|$

So triangle inequality really means that getting rid of cancellation gives you a bigger sum. When you do that for integrals, you get