Math 131B, Wed Sep 02

(Mon = Labor Day) Wed

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 3.3. Reading for ~~Mon~~: 3.4.
- Outline for PS02 due 11pm, completed version due Wed Sep 09.
- Outline for PS03 due Wed Sep 09.
- Next problem session Fri Sep 04, 10:00–noon on Zoom.

Last time

- Extreme Value Theorem (XVT)
- Limits
- Differentiability (chain rule, etc.)

Questions?

& local linearity

Overview of today

$u, v : [a,b] \to \mathbb{R}$

Let $f : [a, b] \to \mathbb{C}$ be bounded, with real and imaginary parts $f(x) = u(x) + iv(x)$. Goal is to define

$$\int_a^b f(x)\,dx.$$

Here is where we really need to start keeping it real, i.e., domain of functions is interval in R.

First, for bounded $v : [a, b] \to \mathbb{R}$, define when

$$\int_a^b v(x)\,dx$$

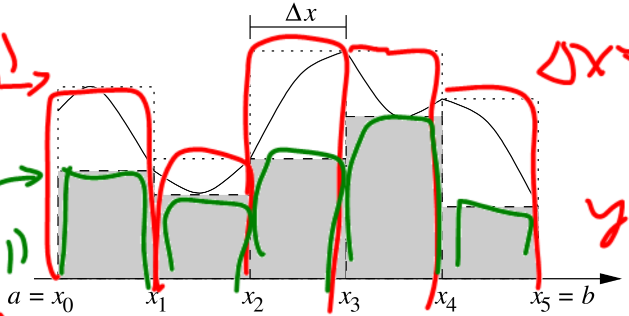
works (**Riemann integrability**) and then use that to define complex version. Then establish technical tools to prove actual examples integrable.

Riemann sums, Calc II style (sort of)

$n=5$

$\Delta x = \frac{b-a}{5}$

$M(v;5,1)$



$m(v;5,1)$

$y=v(x)$

Max $\quad M(v;n,i) = \sup\{v(x) \mid x \in [x_{i-1}, x_i]\}$,

min $\quad m(v;n,i) = \inf\{v(x) \mid x \in [x_{i-1}, x_i]\}$.

rect i

Formula for upper sum? Lower sum?

$U(v;5) = \sum_{i=1}^{5} M(v;5,i)\,\Delta x$ ← upper

$L(v;5) = \sum_{i=1}^{5} m(v;5,i)\,\Delta x$ ← lower

Definition of the Riemann integral, Calc II style (sort of)

Riemann

Upper integral is:

$$\lim_{n\to\infty} \sum_{i=1}^{n} M(v;n,i)\,\Delta x$$

Lower integral is:

$$\lim_{n\to\infty} \sum_{i=1}^{n} m(v;n,i)\,\Delta x$$

$\int_a^b v(x)\,dx$ exists if both upper and lower integrals exist and are equal.

and v is Riemann integrable on [a,b]

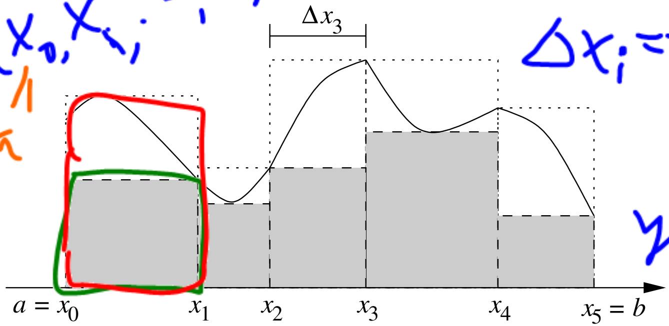
Problem with this setup: If you want to prove stuff about Riemann integral, much better to allow of subdividing [a,b] into uneven subintervals. (For what it's worth, this is also much more practically useful!)

Riemann sums, with uneven partition of region of integration

$b = x_5 \}$ $n = 5$

$P = \{x_0, x_1, \ldots$ $\Delta x_i = x_i - x_{i-1}$

a



$y = v(x)$

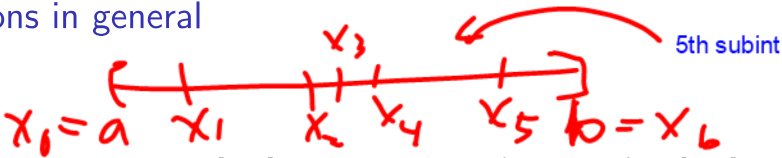
$$M(v; P, i) = \sup \{v(x) \mid x \in [x_{i-1}, x_i]\},$$
$$m(v; P, i) = \inf \{v(x) \mid x \in [x_{i-1}, x_i]\}.$$

Formula for upper sum? Lower sum?

$U(v;P) = \sum_{i=1}^{5} M(v;S,i)\,\Delta x_i$ (upper)

$$L(v;P) = \sum_{i=1}^{5} m(v,S,i)\,\Delta x_i \quad \text{(Lower)}$$

Partitions in general

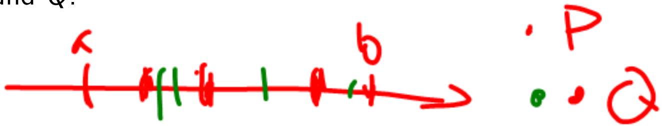


- A **partition** P of $[a, b]$ is a finite subset $\{x_0, \ldots, x_n\} \subset [a, b]$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.
- $[x_{i-1}, x_i]$ is the i**th subinterval** of P.
- $(\Delta x)_i = x_i - x_{i-1}$.

$P \subseteq P \cup Q$
$Q \subseteq P \cup Q$

Suppose P and Q are partitions of $[a, b]$.

- If $P \subseteq Q$, we say that Q is a **refinement** of P.
- $P \cup Q$ is partition of $[a, b]$ obtained from the points of $P \cup Q$, written in ascending order, called the **common refinement** of P and Q.



Definition of the Riemann integral

Let $\mathcal{P}$ be the set of all partitions of $[a, b]$. We define the **upper Riemann integral** and **lower Riemann integral** of v on $[a, b]$ to be

$$\overline{\int_a^b} v(x)\,dx = \inf_{P\in\mathcal{P}} U(v; P),$$
$$\underline{\int_a^b} v(x)\,dx = \sup_{P\in\mathcal{P}} L(v; P).$$

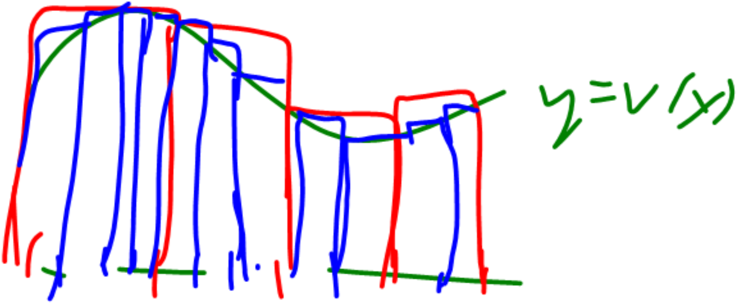
To say that v is **integrable** on $[a, b]$ means that v is bounded on $[a, b]$ and the upper and lower integrals of v on $[a, b]$ are equal. If v is integrable, we define the **Riemann integral of** v **on** $[a, b]$ to be

$$\int_a^b v(x)\,dx = \overline{\int_a^b} v(x)\,dx = \underline{\int_a^b} v(x)\,dx.$$

Why do we define upper integral to be inf of all possible upper Riemann sums?

Ans: When you subdivide, you get a upper sum that should be closer to the "actual area under curve", if it exists. So area under curve should be something like "smallest possible upper sum", i.e., inf of all upper sums.

(Inf here is actually area under green curve.)



Criteria for integrability

Problem: Not obvious that there are any nontrivial examples of integrable functions. (Constant functions not too bad to prove, but even those take a little work.)

So we establish **sequential criteria for integrability**, which we can use to prove, for example, piecewise continuous functions are integrable.

Finer partitions give closer Riemann sums

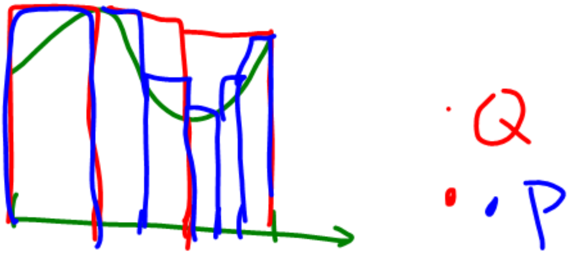
P refines Q

Lemma

Let $v : [a, b] \to \mathbb{R}$ be bounded and let $Q \subseteq P$ be partitions of $[a, b]$. Then

$$L(v; Q) \leq L(v; P) \leq U(v; P) \leq U(v; Q).$$

Picture:



Upper/lower sums squeeze upper/lower integrals

Theorem

Let $v : [a, b] \to \mathbb{R}$ be bounded. Then $\underline{\int_a^b} v(x)\,dx$ and $\overline{\int_a^b} v(x)\,dx$ are both real numbers (and not $\pm\infty$), and for any partition P of $[a, b]$, we have

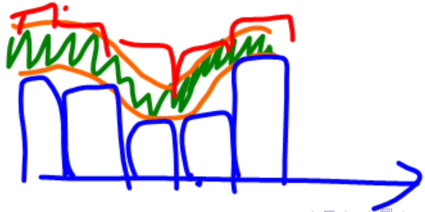
Key pt!

$$L(v; P) \leq \underline{\int_a^b} v(x)\,dx \leq \overline{\int_a^b} v(x)\,dx \leq U(v; P).$$

WTS: Inf of the upper sums is >= sup of the lower sums.

Proof: PS03.

Picture:



Sequential criterion for integrability

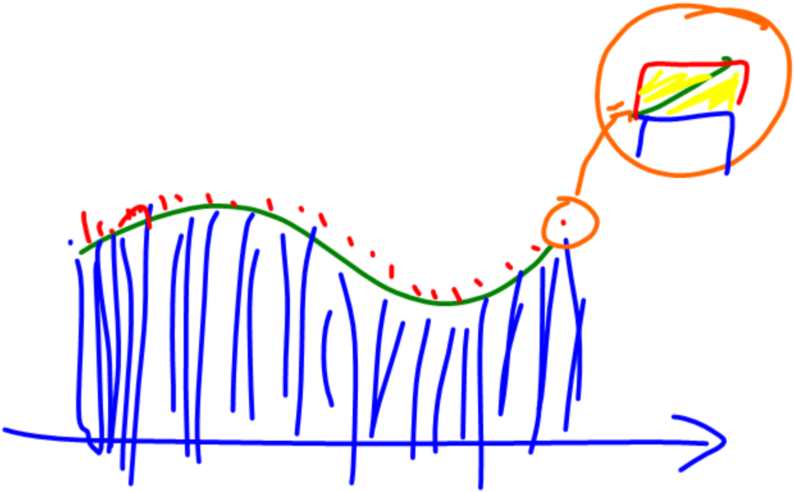
Lemma (Sequential Criteria for Integrability)

Let $v : [a, b] \to \mathbb{R}$ be bounded. Then the following are equivalent.

1. *v is integrable on $[a, b]$.*
2. *There exists a sequence of partitions P_n such that $\lim_{n\to\infty} (U(v; P_n) - L(v; P_n)) = 0$.*
3. *For any $\epsilon > 0$, there exists a partition P such that $U(v; P) - L(v; P) < \epsilon$.*

Furthermore, if condition (2) holds, then

$$\lim_{n\to\infty} L(v; P_n) = \int_a^b v(x)\,dx = \lim_{n\to\infty} U(v; P_n).$$



Cond 3 says that for any epsilon>0, we can pick a partition P such that the sum of the "little yellow areas" between upper and lower rectangles is less than epsilon.

Integrals of complex functions

Definition

Let $f : [a, b] \to \mathbb{C}$ be bounded, and let u and v be the real and imaginary parts of f. To say that f is integrable means that both u and v are integrable, in which case we define

$$\int_a^b f(x)\,dx = \int_a^b u(x)\,dx + i\int_a^b v(x)\,dx.$$

I.e., define integral of a complex function via integral of its real and imaginary parts.

Sequential criterion for complex integrability

Lemma

Let $f : [a, b] \to \mathbb{C}$ be bounded, and for any partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ and $1 \leq i \leq n$, define

$$\mu(f; P, i) = \sup\{|f(x) - f(y)| \mid x, y \in [x_{i-1}, x_i]\},$$

$$E(f; P) = \sum_{i=1}^{n} \mu(f; P, i)(\Delta x)_i.$$

Then the following are equivalent.

1. *f is integrable on $[a, b]$.*
2. *For any $\epsilon > 0$, there exists a partition P such that $E(f; P) < \epsilon$.*