Math 131B, Mon Aug 31

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 3.1-3.2. Reading for Wed: 3.3.
- Outline for PS02 now due **Wed Sep 02**. Complete due Wed Sep 09.
- Next problem session Fri Sep 04, 10:00–noon on Zoom.

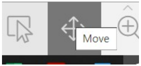
How to use Limnu

Limnu is the online whiteboard software we'll use to collaborate during problem sessions, office hours, and class.

- Each day we'll start with a new board, sometimes preloaded with materials. The board will have an address of the form:

  `http://go.limnu.com/random-words`

  The board will usually be shared as a clickable link, either in chat or in an email before problem sessions.
- Click on the link or type the address into a browser on a machine where you have a touchscreen (e.g., smartphone or tablet). If this is your first time using limnu, you may have to set up an account first.
- Draw and write! And by default, stay in "Move" mode:



Last time

Results about metric spaces and continuity, including (PS02):

Theorem (Bolzano-Weierstrass in $\mathbb{C}$)

Every bounded sequence in $\mathbb{C}$ has a convergent subsequence.

Extreme Value Theorem (XVT)

Theorem

Let X be a closed and bounded subset of $\mathbb{C}$, and let $f : X \to \mathbb{R}$ be continuous. Then f attains both an absolute maximum and an absolute minimum on X; that is, there exist $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.

Need real outputs for this to make sense.

Proof: Argument has two parts.

1. First show that f must be bounded.
2. Then show that f attains the sup of its values (i.e., max).

PS02

Both parts use:

- B-W on $\mathbb{C}$
- If X is a closed subset of $\mathbb{C}$, and x_n is a convergent sequence in X, then $\lim_{n\to\infty} x_n$ is still in X.

Proof of boundedness part of XVT

ABC = Assume By way of Contradiction

X cl, bd in $\mathbb{C}$, $f: X \to \mathbb{R}$, cont.

WTS that f is a bounded function. By contradiction.

ABC: f is not a bounded function. (f bounded = exists M such |f(x)| < M for all x in X)

So, for every integer n>0, there exists some $x_n \in X$ s.t. $|f(x_n)| > n$.

(n not a bd for $|f(x)|$.)

X bd, so x_n is bd seq.

B-W $\Rightarrow \exists$ conv subseq x_{n_k}. So

$\lim_{k \to \infty} x_{n_k} = c \in X$ b/c X closed.

On one hand

$f(c) = f\left(\lim_{k\to\infty} x_{n_k}\right)$ (defn c)

$= \lim_{k\to\infty} f(x_{n_k})$, (seq cont)

$f(\lim -) = \lim f(-)$

so $f(x_{n_k})$ is conv seq.

OTOH, $f(x_{n_k})$ is not bd, b/c

$|f(x_{n_k})| > n_k \to \infty$.

$\boxed{\text{CONTRA}}$

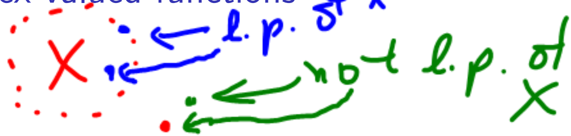
Techniques that will be useful to you in the future:

1. If you can get a bounded seq in R or in C, B-W gives conv subsequence

2. Note the notation for convergent subsequences: $\lim_{k \to \infty} x_{n_k} = L$

3. f continuous means f (lim blah) = lim f(blah).

Limits of complex-valued functions



Definition

$X \subseteq \mathbb{C}$ nonempty. To say that a is a **limit point** of X means that there exists z_n in X such that $\lim_{n\to\infty} z_n = a$ and $\underline{z_n \neq a}$ for all n.

Definition (Need seqs in X that approach a but never reach it.)

$X \subseteq \mathbb{C}$ nonempty, $f : X \to \mathbb{C}$ be a function, and let a be a limit point of X. To say that $\lim_{z\to a} f(z) = L$ means that one of the following conditions holds:

- (**Sequential limit**) For every sequence z_n in X such that $\lim_{n\to\infty} z_n = a$ $\boxed{\text{and } z_n \neq a \text{ for all } n,}$ we have that $\lim_{n\to\infty} f(z_n) = L$.
- (ϵ-δ **limit**) For every $\epsilon > 0$, there exists some $\delta(\epsilon) > 0$ such that if $|z - a| < \delta(\epsilon)$ $\boxed{\text{and } z \neq a,}$ then $|f(z) - L| < \epsilon$.

If we delete stuff in red boxes, we get continuity.

Limit laws work like laws of continuity

Limit of the sum is the sum of the limits, etc.

g sa

One new wrinkle:

Lemma (Squeeze Lemma)

$X \subseteq \mathbb{C}$ nonempty, $f, g, h : X \to \mathbb{R}$ such that $f(z) \leq g(z) \leq h(z)$ for all $x \in X$, and for some limit point a of X, suppose

$$\lim_{z\to a} f(z) = L = \lim_{z\to a} h(z).$$

Then $\lim_{z\to a} g(z) = L$.

Proof on PS02.

Differentiation

We need defn of lim of fn because the domain of the difference quotient does not include a.

$X \subseteq \mathbb{C}$ such that every point of X is a limit point.

Definition

$f : X \to \mathbb{C}$, $a \in X$. To say that f is **differentiable** at a means

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists (where $h = z - a$). To say that f is **differentiable on** X means that for all $a \in X$, f is differentiable at a; and to say that f is **continuously differentiable on** X means that f is differentiable on X and $f' : X \to \mathbb{C}$ is continuous.

I.e., it's 1-variable calculus! (But the variable is complex now.) And definition used in same way as in 1-variable real calculus.

First laws of calculus work as before

E.g., if f is differentiable at $a \in X$, then f is continuous at a. Also:

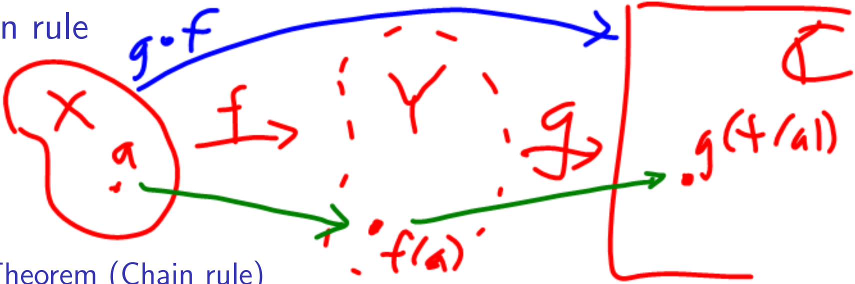
"differentiability implies continuity"

Theorem

$f, g : X \to \mathbb{C}$ are differentiable at a. Then:

1. *For $c \in \mathbb{C}$, cf is differentiable at a, with derivative $(cf)'(a) = cf'(a)$.* const mult
2. *$f + g$ is differentiable at a, with derivative $(f+g)'(a) = f'(a) + g'(a)$.* sum
3. *$\overline{f}$ is differentiable at a, with derivative $\overline{f}'(a) = \overline{f'(a)}$.* conj rule
4. *fg is differentiable at a, with derivative $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.* prod rule
5. *If $g(z) \neq 0$ for all $z \in X$, then f/g is differentiable at a, with derivative* $$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{g(a)^2}.$$ quot rule

Chain rule



Theorem (Chain rule)

If $f : X \to Y$ diff at a and $g : Y \to \mathbb{C}$ diff at $f(a)$, then $g \circ f : X \to \mathbb{C}$ diff at a, and $(g \circ f)'(a) = g'(f(a))f'(a)$.

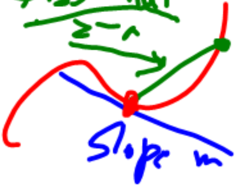
Best proved using **local linearity**.

$$g'(\underset{\substack{\uparrow \\ f(a)}}{\quad})\,f'(\underset{\substack{\uparrow \\ a}}{\quad})$$

Local linearity

$f : X \to \mathbb{C}$ $a \in X$

slope $\frac{f(z)-f(a)}{z-a}$



Lemma

TFAE:

some tangent slope m exists s.t.

- *f diff at a.*
- *There exists some $m \in \mathbb{C}$ such that if we define*

sec slope

error in sec slopes

$$\boxed{E_f(z)} = \begin{cases} \boxed{\dfrac{f(z)-f(a)}{z-a}} - m & \text{for } z \neq a, \\ 0 & \text{for } z = a, \end{cases}$$

for all $z \in X$, then $E_f(z)$ is continuous at a (i.e., $\lim_{z \to a} E_f(z) = 0$).

Futhermore, if either (and therefore both) of these conditions hold, $m = f'(a)$.

Local linearity (most common special case)

Use this to prove complex valued chain rule in PS02.

Corollary (Local Linearity)

If $f : X \to \mathbb{C}$ diff at $a \in X$, then there exists $E_f : X \to \mathbb{C}$ such that E_f is continuous at a, $E_f(a) = 0$, and

$$\boxed{f(z) = f(a) + (f'(a) + E_f(z))(z - a)}$$

for all $z \in X$. □

$$f(z) \approx f(a) + f'(a)(z-a)$$

Diff means: Local linear approximation to f(z) has a slope error E_f(z) that goes to 0 as z -> a.

Mean Value Theorem

Theorem (Mean Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ be a real-valued function that is continuous on $[a, b]$ and differentiable on (a, b). Then there exists some $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c). \quad \square$$

Only works for **real-valued** functions. But it does have complex-valued consequences, e.g.:

Corollary (Zero Derivative Theorem)

Let X be a path-connected subset of $\mathbb{C}$, and let $f : X \to \mathbb{C}$ be a function. Suppose either that $f'(z) = 0$ for all $x \in X$, or $X = [a, b]$, f is continuous on $[a, b]$, and $f'(x) = 0$ for all $x \in (a, b)$. Then f is constant on X.