Math 131B, Wed Aug 26

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 2.5, 3.1. Reading for Mon: 3.1-3.2.
- PS01 due tonight at 11pm; outline for PS02 due Mon Aug 31.
- Next problem session Fri Aug 28, 10:00–noon on Zoom.

Last time

1.1.2) $x = \frac{5}{13}$

$\cos\left(2\pi n \left(\frac{5}{13}\right)\right) \stackrel{?}{=} 1$

ans: list of ∞ n to put in here to get 1

Ended with:

- Limit of a sequence in $\mathbb{C}$
- Limit of a sequence in a metric space

Questions?

2.1.5:

A) 1 A) 2

C) 2 C) 1

Dense subsets of a metric space

(2.4)

Let X be a metric space and Y a subset of X. Then it can be shown that the following conditions are equivalent:

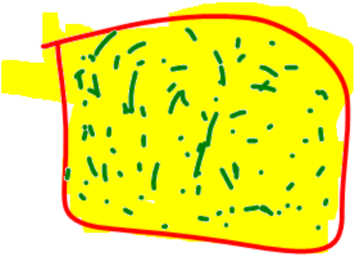
1. For every $x \in X$ and every $\epsilon > 0$, there exists some $y \in Y$ such that $d(x, y) < \epsilon$. "Every point of X has a point of Y arb close"
2. For every $x \in X$, there exists some sequence y_n in Y such that $\lim_{n\to\infty} y_n = x$. Every point of X is the limit of a seq in Y

Definition

To say that a subset Y of a metric space X is **dense** in X means that either (and therefore, both) of the above conditions hold.

Example

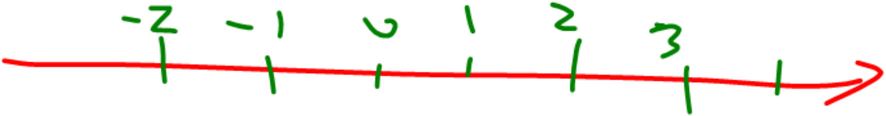
The rationals $\mathbb{Q}$ are a dense subset of the metric space $\mathbb{R}$.



Picture of a dense subset:

X is the underlying space
Y is a kind of "dust" that doesn't take up much area with in X, but is still everywhere.

(Dense subsets will be important to us later when constructing approximations.)



Z is not a dense subset of R, but Q is.
(Rational numbers include any finite decimal, so this is related to fact that any real can be approx arb closely by a finite decimal.)

Cauchy sequences in a metric space (2.5)

Need the following idea in a metric space X to replace order completeness:

"Defn". To say that a_n in X is **Cauchy** means that the points of a_n get closer to each other, instead of closer to some known limit L. I.e.:

$\forall \epsilon > 0$ No matter how epsilon close we require

eventually

$\exists N(\epsilon)$ s.t.

If $n, k > N(\epsilon)$

then $d(a_n, a_k) < \epsilon$

the terms of the sequence a_n will be that close to each other.

(So it appears that a_n is converging to some limit, even if we don't know what that limit is.)

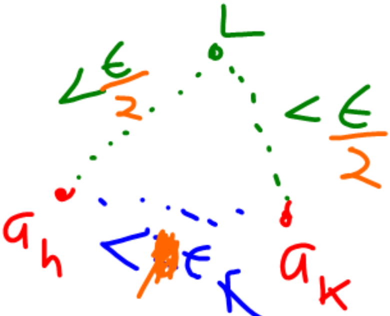
Convergent implies Cauchy

Theorem

Let a_n be a convergent sequence in a metric space X. Then a_n must be Cauchy.

the a_n eventually get very close to L

Why. Suppose $\lim_{n\to\infty} a_n = L$. Then:



$n, k > N\left(\frac{\epsilon}{2}\right)$

by triangle inequality

C: Eventually, the a_n must get very close to each other.

Bolzano-Weierstrass and Cauchy completeness

I.e., completeness means that any sequences that behaves like a convergent sequence actually converges to a point of X.

Definition

To say that a metric space X is **Cauchy complete**, or simply **complete**, means that any Cauchy sequence in X converges to some limit in X.

This is a Big Deal in Analysis I:

Theorem (Bolzano-Weierstrass)

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

PS02: Use Bolzano-Weierstrass in $\mathbb{R}$ to prove same, but for $\mathbb{C}$.

Example of a non-complete space: Take R, delete 0.

$\mathbb{C}$ is a complete metric space

$n_1 < n_2 < n_3 < \cdots$

∞ seq of indices

Suppose we know B-W in $\mathbb{C}$ (PS02). Then:

Corollary

The complex numbers are a complete metric space.

Sketch of proof.

A. Suppose a_n is a Cauchy sequence in C (complex numbers).

Hard part: Create $L \in \mathbb{C}$.

B-W for $\mathbb{C}$ $\Rightarrow$ $\exists$ conv subseq

$a_{\boxed{n_k}}$ s.t. $\lim_{k\to\infty} a_{n_k} = L \in \mathbb{C}$.

Then, since a_{n_k} converges to L and the terms of a_n are all eventually close to each other, the terms of a_n converge to L

details!

C. a_n converges to some L in C.

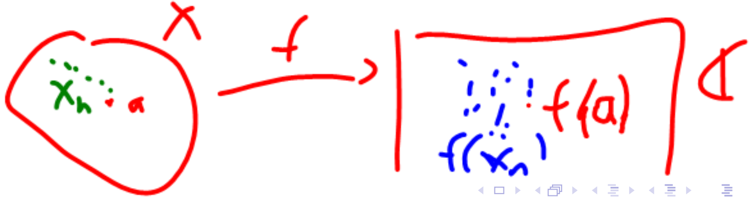
Defn of continuity for $f : \mathbb{C} \to \mathbb{C}$

(Eventhal: $f : \mathbb{R} \to \mathbb{C}$)

Definition

Let X be a nonempty subset of $\mathbb{C}$, let $f : X \to \mathbb{C}$ be a function, and let a be a point in X. To say that f is **continuous** at a means that one of the following conditions holds:

- (**Sequential continuity**) For every sequence x_n in X such that $\lim_{n \to \infty} x_n = a$, we have that $\lim_{n \to \infty} f(x_n) = f(a)$.
- (ϵ-δ **continuity**) For every $\epsilon > 0$, there exists some $\delta(\epsilon) > 0$ such that if $|x - a| < \delta(\epsilon)$, then $|f(x) - f(a)| < \epsilon$.



Defn of continuity for $f : X \to Y$

Definition

Let X and Y be metric spaces, let $f : X \to Y$ be a function, and let a be a point in X. To say that f is **continuous** at a means that one of the following conditions holds:

- (**Sequential continuity**) For every sequence x_n in X such that $\lim_{n\to\infty} x_n = a$, we have that $\lim_{n\to\infty} f(x_n) = f(a)$.
- (ϵ-δ **continuity**) For every $\epsilon > 0$, there exists some $\delta(\epsilon) > 0$ such that if $d(x, a) < \delta(\epsilon)$, then $d(f(x), f(a)) < \epsilon$.

To say that f is **continuous on** X means that f is continuous at a for all $a \in X$.

Equivalence of sequential and ϵ-δ continuity

Theorem

Let X and Y be metric spaces, let $f : X \to Y$ be a function, and let a be a point in X. Then f is sequentially continuous at a if and only if f is ϵ-δ continuous at a.

See book for proof.

Laws of continuity (from calculus)

Theorem

Let X be a subset of $\mathbb{C}$, let $f, g : X \to \mathbb{C}$ be functions, and for some $a \in X$, suppose that f and g are contiuous at a. Then:

1. *For $c \in \mathbb{C}$, $cf(x)$ is continuous at a.*
2. *$f(x) + g(x)$ is continuous at a.*
3. *$\overline{f(x)}$ is continuous at a.*
4. *$f(x)g(x)$ is continuous at a.*
5. *If $g(x) \neq 0$ for all $x \in X$, then $f(x)/g(x)$ is continuous at a.*

All follow from seq cont
+ limit laws for seqs.

Theorem

Let X, Y, and Z be metric spaces, let $f : X \to Y$ and $g : Y \to Z$ be functions, let a be a point in X, and suppose that f is continuous at a and g is continuous at $f(a)$. Then $g \circ f$ is continuous at a.

Pf on PS02

Uniform continuity

only ϵ-δ

Definition

Let X be a nonempty subset of $\mathbb{C}$ and let $f : X \to \mathbb{C}$ be a function. To say that f is **uniformly continuous** on X means that for every $\epsilon > 0$, there exists some $\delta(\epsilon) > 0$ such that if $x, y \in X$ and $|x - y| < \delta(\epsilon)$, then $|f(x) - f(y)| < \epsilon$.

Point is that $\delta(\epsilon)$ no longer depends on point of continuity (i.e., no longer $\delta(\epsilon, a)$) which is what you get for f continuous at all $a \in X$.

Key fact is:

Theorem

If X is a closed and bounded subset of $\mathbb{C}$ and $f : X \to \mathbb{R}$ is continuous, then f is uniformly continuous on X.

(Another miracle of B-W!)

Extreme Value Theorem (XVT)

Theorem

Let X be a closed and bounded subset of $\mathbb{C}$, and let $f : X \to \mathbb{R}$ be continuous. Then f attains both an absolute maximum and an absolute minimum on X; that is, there exist $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.

Proof: Argument has two parts:

1. First show that f must be bounded.
2. Then show that f attains the sup of its values (i.e., max).

Both parts use:

- B-W on $\mathbb{C}$
- If X is a closed subset of $\mathbb{C}$, and x_n is a convergent sequence in X, then $\lim\limits_{n\to\infty} x_n$ is still in X.

Proof of boundedness part of XVT