# Math 131B, Mon Aug 24

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 2.2–2.4. Reading for ~~Mon~~ Wed: 2.5, 3.1.
- PS00, PS01 due tonight 11pm; PS01 due Wed Aug 26.
- Next problem session Fri Aug 28, 10:00–noon on Zoom.

outline full

# ACC and Sup Inequality

## Theorem (Arbitrarily Close Criterion)

S=red

*Suppose $S$ is a nonempty subset of $\mathbb{R}$, and suppose $u$ is an upper bound for $S$. Then the following are equivalent:*

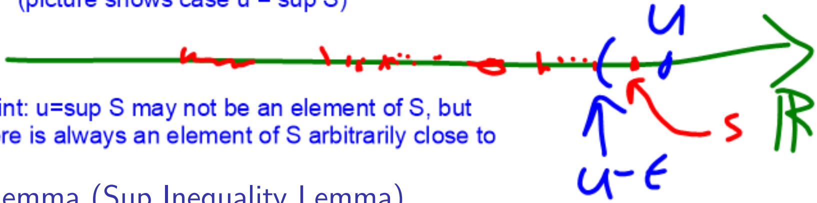
1. *For every $\epsilon > 0$, there exists some $s \in S$ such that $u - s < \epsilon$.*
2. $u = \sup S$.

Pf PS01

(i.e., s has dist < $\epsilon$ from u)

Picture:

(picture shows case u = sup S)




Point: u=sup S may not be an element of S, but there is always an element of S arbitrarily close to u.

## Lemma (Sup Inequality Lemma)

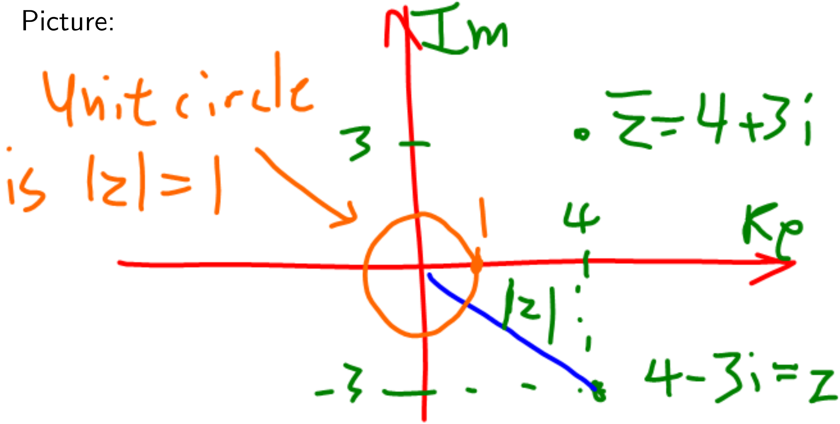
*If $S$ is a nonempty bounded subset of $\mathbb{R}$, then $\sup S \leq u$ if and only if $u$ is an upper bound for $S$.*

Point: Method for proving ineqs w/ sup.

# The complex numbers $\mathbb{C}$

Are polynomials in the variable $i$ with real coefficients, with the relation $i^2 = -1$.
(Actually the fancy grownup definition of $\mathbb{C}$)

Picture:



# Absolute value and conjugates

For $z = a + bi$ in $\mathbb{C}$, define:

$$|z| = \sqrt{a^2 + b^2}, \qquad \overline{a + bi} = a - bi$$

Lots of formulas that result from that and brute force; most frequently used include (for $z, w \in \mathbb{C}$):

$$|z|^2 = z\overline{z}, \qquad |zw| = |z|\,|w|$$

Go check these yourself!

# Metrics

Metrics define a "distance" $d$ on a set $X$:

## Definition

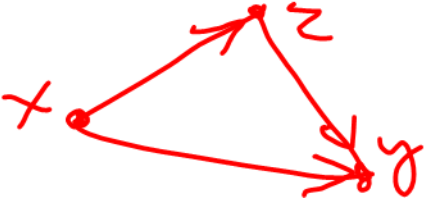
A **metric** on $X \neq \emptyset$ is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$, we have:

nonneg

distinct points have nonzero distance

1. $d(x, y) \geq 0$.
2. $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, y) = d(y, x)$.
4. (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$.

distance is symmetric




(Can't shorten distance from Fremont to San Jose by going through Milpitas)

# Metric spaces

A **metric space** is $(X, d)$ where $X$ is a set and $d$ is a particular metric on $X$. (Often omit $d$.)

Turns out much of analysis I can be generalized from $f : \mathbb{R} \to \mathbb{R}$ to $f : X \to Y$, where $X$ and $Y$ are metric spaces.

the calc metric

Example: $X = \mathbb{R}$, $d(x, y) = |x - y|$.

Ex of metric spaces

Example: $X = \mathbb{C}$, $d(z, w) = |z - w|$. (Proof: PS01 and other problems in 2.3.)

(that |z-w| is a metric)

To prove $d(z,w)=|z-w|$ is a metric on C, need to show:

1. $|z-w| >= 0$

2. $|z-w| = 0$ if and only $z=w$

3. $|z-w| = |w-z|$

4. Triangle inequality. (This is the interesting part -- see Probs. 2.3.1 and 2.3.2.)

# Sequences and limits

(often: $N = \mathbb{N}$)

## Definition

A **sequence** in a set $X$ is a function $a : N \to X$, where $N$ is all integers $\geq$ some starting point (usually 0 or 1).

Usually write $a_n$ instead of $a(n)$. $a_1\, a_2\, a_3 \ldots \in X$

Subsequences work as in analysis I (see text for notation).

**Defn:** For a complex-valued sequence $a_n$ and $L \in \mathbb{C}$, to say that $\lim_{n\to\infty} a_n = L$ means that:

$\forall \epsilon > 0$ — No matter how epsilon-close we want the values of a_n to be to L,

$\exists N(\epsilon)$ s.t.

eventually they will always be

If $n \in \mathbb{Z}$, $n > N(\epsilon)$

then $\underbrace{|a_n - L| < \epsilon}_{\text{closer than that epsilon.}}$

# Let's redo Analysis I

So now we can take the beginning of analysis I, like the limit laws for sequences, and redo it, replacing $\mathbb{R}$ with $\mathbb{C}$. Example:

"If" = A

## Theorem

*Let $a_n$ and $b_n$ be sequences in $\mathbb{C}$, and suppose that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then $\lim_{n\to\infty}(a_n + b_n) = L + M$.*

"then" = C

"The limit of the sum is the sum of the limits"

A. a_n, b_n seqs in C, lim a_n = L, lim b_n = M.

$\forall \epsilon_1 > 0, \exists N_1(\epsilon_1)$ s.t. if $n > N_1(\epsilon_1)$ then $|a_n - L| < \epsilon_1$

$\forall \epsilon_2 > 0, \exists N_2(\epsilon_2)$ s.t. if $n > N_2(\epsilon_2)$, $|b_n - M| < \epsilon_2$

$\boxed{A}\ \epsilon > 0$

Choose $N(\epsilon) =$

$\boxed{A_3}$ $n > N(\epsilon)$

choose this

to ensure

$$\underbrace{|(a_n - L) + (b_n - M)| \le \overbrace{|a_n - L|}^{<\frac{\epsilon}{2}} + \overbrace{|b_n - M|}^{<\frac{\epsilon}{2}}}_{}$$

$< \epsilon$

So $|(a_n - L) + (b_n - M)| < \epsilon$

$C_3$ $|(a_n + b_n) - (L+M)| < \epsilon$.

$C_2$ If $n > N(\epsilon)$, then $|(a_n+b_n)-(L+M)| < \epsilon$

$C_1$ $\exists N(\epsilon)$ s.t. if $n > N(\epsilon)$, then...

$\leftarrow$ if

$\forall \epsilon > 0$

then

$\exists N(\epsilon)$ s.t.

If $n > N(\epsilon)$

then $|(a_n+b_n)-(L+M)| < \epsilon$.

C. lim (a_n + b_n) = L + M.

# Closed subsets of $\mathbb{C}$

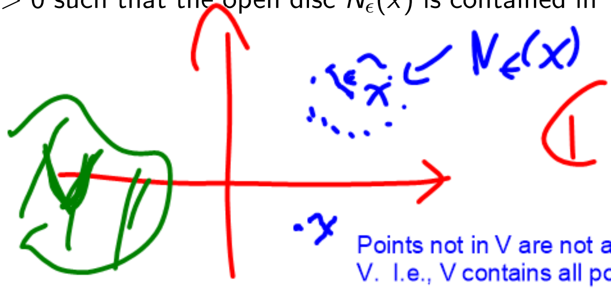
For $x \in \mathbb{C}$ and $r \geq 0$, the **open disc of radius $r$ around $x$** is

$$N_r(x) = \{y \in \mathbb{C} \mid |y - x| < r\}\,.$$

## Definition

Let $V$ be a subset of $\mathbb{C}$, and let $V^c = \mathbb{C} - V$ be the **complement** of $V$ in $\mathbb{C}$.
To say that $V$ is **closed** in $\mathbb{C}$ means that for every $x \in V^c$, there exists some $\epsilon > 0$ such that the open disc $N_\epsilon(x)$ is contained in $V^c$.

Picture:




Points not in V are not almost ir
V. I.e., V contains all points
that are almost in V.

# Limits in general metric spaces

**Defn:** For a sequence $a_n$ in a metric space $X$ and $L \in X$, to say that $\lim_{n\to\infty} a_n = L$ means:

$\forall \epsilon > 0$

$\exists N(\epsilon)$ s.t.

If $n \in \mathbb{Z}$, $n > N(\epsilon)$

then $d(a_n, L) < \epsilon$

eventually

This generalizes defn of limit of a real-valued seq and defn of limit of a C-valued seq.

# Squeeze Lemma

## Lemma (Metric Squeeze Lemma)

*Let $x_n$ be a sequence in a metric space $X$, and suppose that for some $L \in X$, there exists a sequence $d_n$ in $\mathbb{R}$ such that $d(x_n, L) < d_n$ for all $n$ and $\lim_{n\to\infty} d_n = 0$. Then $\lim_{n\to\infty} x_n = L$.*

Proof: PS01. (Point is for you to get practice in using the definition of the limit of a sequence in a metric space.)

# Dense subsets of a metric space

Let $X$ be a metric space and $Y$ a subset of $X$. Then it can be shown that the following conditions are equivalent:

1. For every $x \in X$ and every $\epsilon > 0$, there exists some $y \in Y$ such that $d(x, y) < \epsilon$.
2. For every $x \in X$, there exists some sequence $y_n$ in $Y$ such that $\lim_{n \to \infty} y_n = x$.

## Definition

To say that a subset $Y$ of a metric space $X$ is **dense** in $X$ means that either (and therefore, both) of the above conditions hold.

## Example

The rationals $\mathbb{Q}$ are a dense subset of the metric space $\mathbb{R}$.