Math 131B, Wed Dec 02

Colloquium 3pm today: Stephanie Salomone, "What I Believe"
Especially of interest to future teachers!

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 12.4. THE END
- Otline for PS11 due tonight; full version due Mon Dec 07.
- Problem session, Fri Dec 04, 10:00am–noon on Zoom. PS10 & 11
- **FINAL EXAM, MON DEC 14, 9:45am–noon.** 4.8.6 xtra cred

Recap

Laplace transform
FT for C

Schwartz

Definition

$\mathcal{S}(\mathbb{R})$ is the space of all $f : \mathbb{R} \to \mathbb{C}$ such that for all $k \geq 0$, the kth derivative $f^{(k)}(x)$ of f exists for all $x \in \mathbb{R}$ and is rapidly decaying.

Definition

For $f \in \mathcal{S}(\mathbb{R})$, define the **Fourier transform** of f to be the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ given by

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \gamma x} \, dx$$

for any $\gamma \in \mathbb{R}$. γ = freq

Note that because we now assume $f \in \mathcal{S}(\mathbb{R})$, integral definitely converges.

Last ~~time~~ week

$(f, g \in S(\mathbb{R}))$

- Convolution $f * g : \mathbb{R} \to \mathbb{C}$ defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - t) g(t)\, dt.$$

- Dirac kernel $K_t : \mathbb{R} \to \mathbb{R}$ $(t \in \mathbb{R},\ t > 0)$:
  - For all $t > 0$ and all $x \in \mathbb{R}$, $K_t(x) \geq 0$;
  - For all $t > 0$, $\displaystyle\int_{-\infty}^{\infty} K_t(x)\, dx = 1$; and

    Prob dist.
  - For fixed $\eta > 0$, we have $\displaystyle\lim_{t \to 0^+} \int_{|x| \geq \eta} K_t(x)\, dx = 0.$

    conc at 0
- Thm: $\displaystyle\lim_{t \to 0^+} (f * K_t)(x) = f(x).$ ✓
- Example of a Dirac kernel: Gauss kernel ✓

$$G_t(x) = \frac{1}{t} \exp\left(\frac{-\pi x^2}{t^2}\right).$$

PS10, Prob 4.8.3

Properties of Fourier transform

	Function (in x)	Fourier transform (in γ)
1	$f(x+a)$	$e^{2\pi i a\gamma}\hat{f}(\gamma)$
2	$e^{2\pi i a x}f(x)$	$\hat{f}(\gamma-a)$
3	$f(-x)$	$\hat{f}(-\gamma)$

In operator notation:

$$U(f)=\hat{f} \qquad (s_{-1}(f))(x)=f(-x)$$

$$(\tau_a(f))(x)=f(x+a) \qquad (\mu_a(f))(x)=e^{2\pi i a x}f(x)$$

The above table says:

1 $$U(\tau_a(f))=\mu_a(U(f))$$

2 $$U(\mu_a(f))=\tau_{-a}(U(f))$$

3 $$U(s_{-1}(f))=s_{-1}(U(f))$$

I.e., $U\tau_a=\mu_a U$, $U\mu_a=\tau_{-a}U$, $Us_{-1}=s_{-1}U$.

Recall: How to write Fourier transform with freqency variable same as the variable you started with (time variable)

$$f(x) \in \mathcal{S}(\mathbb{R})$$

$$\rightarrow \hat{f}(x) = \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i x y}\, dy$$

Then $U : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$

$$U(f) = \hat{f}$$

"Pass the hat" and the Gauss kernel

Theorem Pass the Hat

If $f, g \in \mathcal{S}(\mathbb{R})$, then $\displaystyle\int_{-\infty}^{\infty} \hat{f}(x)g(x)\,dx = \int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx.$

Theorem $\boxed{\text{PS10}}$

The Fourier transform of $f(x) = e^{-\pi x^2}$ is $\hat{f}(\gamma) = e^{-\pi\gamma^2}$. In other words, f is its own Fourier transform, or $U(f) = f$.

More generally, for $t > 0$, let $G_t(x) = \dfrac{1}{t}\exp\left(\dfrac{-\pi x^2}{t^2}\right)$ *be the Gauss kernel. Then*

$$\hat{G}_t(\gamma) = e^{-\pi t^2\gamma^2}, \qquad U(U(G_t)) = \hat{\hat{G}}_t = G_t.$$

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1$$

An ugly lemma

$\hat{\hat{h}}_x$

const var

Lemma

For $f \in \mathcal{S}(\mathbb{R})$ and constant $x \in \mathbb{R}$, let $h_x(y) = f(-x-y)$. Then $\hat{\hat{f}}(x-y) = \hat{\hat{h}}_x(y)$, where the Fourier transform is calculated in the variable y.

$U(U(f))$, then plug $x-y$.

For clarity, $\hat{\hat{h}}_x$ means $U(U(h_x))$.

Proof: First get $h_x(y)$ by applying operators to $f(y)$:

$$h_x(y) = f(-x-y) = f(-(y+x))$$

$$= (S_{-1}(f))(y+x) = \tau_x(S_{-1}(f))(y)$$

So $h_x = \tau_x S_{-1} f$.

Then get $\hat{\hat{f}}(x-y)$ by applying operators to $f(y)$:

$$\hat{f}(x-y) = (U(U(f)))(x-y) = (U(U(f)))(-(y-x))$$

$$= (S_{-1}(U(U(f))))(y-x) = (\tau_{-x} S_{-1} U U(f))(y)$$

Then previous operator facts give:

$h_x = \tau_x s_{-1} f$

other: $\tau_{-x} s_{-1} \mathcal{U}\mathcal{U} f$

$$\mathcal{U}\mathcal{U}(h_x)$$
$$= \mathcal{U}\mathcal{U}\tau_x s_{-1}(f)$$
$$= \mathcal{U}\mu_x \mathcal{U} s_{-1}(f)$$
$$= \tau_{-x}\mathcal{U}\mathcal{U} s_{-1}(f)$$
$$= \tau_{-x} s_{-1}\mathcal{U}\mathcal{U}(f),$$ the fn w/formula is $\bar{f}(x-y)$.

Inversion Theorem in $\mathcal{S}(\mathbb{R})$

$(s_{-1}f)(x)$

Theorem

For $f \in \mathcal{S}(\mathbb{R})$, we have that

$$\hat{\hat{f}}(x) = \int_{-\infty}^{\infty} \hat{f}(\gamma) e^{-2\pi i \gamma x}\, d\gamma = f(-x). \tag{1}$$

$$\boxed{\text{WTS } \hat{\hat{f}} = s_{-1} f}$$

Substituting $-x$ for x, we get:

$$\underbrace{\int_{-\infty}^{\infty} \hat{f}(\gamma) e^{2\pi i \gamma x}\, d\gamma}_{\text{inverse FT}} = f(x)$$

Proof of inversion

Define

$$g(x) = f(-x) = (s_{-1}(f))(x)$$
$$h_x(y) = f(-x-y) = g(x+y)$$

$g = s_{-1} f$

WTS $\hat{\hat{f}} = g$

Then for any fixed $t > 0$, we have

$$(\hat{\hat{f}} * G_t)(x) = \int_{-\infty}^{\infty} \hat{\hat{f}}(x-y)\, G_t(y)\, dy$$

UL

$$= \int_{-\infty}^{\infty} \hat{\hat{h}}_x(y)\, G_t(y)\, dy$$

$(\text{Pass hat})^2$

$$= \int_{-\infty}^{\infty} h_x(y)\, \hat{\hat{G}}_t(y)\, dy$$

$$= \int_{-\infty}^{\infty} g(x+y)\, \overbrace{G_t(y)}^{f}\, dy \qquad \checkmark\ \widehat{G_t} = G_t$$

$u = -y$

$du = -dy$

$$= \int_{\infty}^{-\infty} g(x-u)\, \underbrace{G_t(-u)}_{G_t(u)}\, (-du)$$

G_t even fn

$$= \int_{-\infty}^{\infty} g(x-u)\, G_t(u)\, du$$

$$= \int_{-\infty}^{\infty} g(x-y)\, G_t(y)\, dy$$

$$= (g * G_t)(x)$$

So: $\hat{\hat{f}} = \lim\limits_{t\to 0^+} \hat{\hat{f}} * G_t = \lim\limits_{t\to 0^+} g * G_t = g$

Isomorphism Theorem in $\mathcal{S}(\mathbb{R})$

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$$

Theorem

For $f, g \in \mathcal{S}(\mathbb{R})$, we have that $\langle f, g\rangle = \left\langle \hat{f}, \hat{g} \right\rangle$. *In particular,* $\|f\| = \left\|\hat{f}\right\|$.

$$\int_{-\infty}^{\infty} \hat{f}(\gamma)\overline{\hat{g}(\gamma)}\,d\gamma$$

Compare isomorphism theorem for $L^2(S^1)$:

$\boxed{\text{PS II}}$

?

Theorem

For $f, g \in L^2(S^1)$, we have that

$$\int_0^1 f\bar{g}\,dx = \langle f, g\rangle = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}.$$

∞ dot prod!

If we define $\ell^2(\mathbb{Z})$ to be the space of all two-sided $a(n)$ such that $\sum_{n\in\mathbb{Z}} |a(n)|^2 < \infty$, then above RHS is $\left\langle \hat{f}, \hat{g} \right\rangle$.