Math 131B, ~~Wed Nov 18~~

Mon Nov 30

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 12.3. Reading for Wed Dec 02: 12.4.
- PS10 due tonight; outline for PS11 due Wed Dec 02.
- Problem session, Fri Dec 04, 10:00am–noon on Zoom.
- **FINAL EXAM, MON DEC 14.**

9:45–noon

Recap

Definition

$\mathcal{S}(\mathbb{R})$ is the space of all $f : \mathbb{R} \to \mathbb{C}$ such that for all $k \geq 0$, the kth derivative $f^{(k)}(x)$ of f exists for all $x \in \mathbb{R}$ and is rapidly decaying.

Definition

For $f \in \mathcal{S}(\mathbb{R})$, define the **Fourier transform** of f to be the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ given by

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \gamma x}\, dx$$

for any $\gamma \in \mathbb{R}$.

Note that because we now assume $f \in \mathcal{S}(\mathbb{R})$, integral definitely converges.

$n \in \mathbb{Z}$

Compare FS: $\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x}\, dx$

	f domain	$\hat{f}$ domain
F.S.	S^1	$\mathbb{Z}$
F.T.	$\mathbb{R}$	$\mathbb{R}$

See: Poisson summation

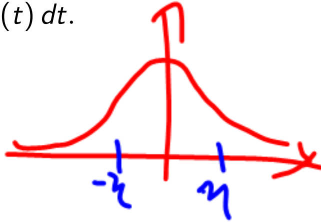
Laplace	$\mathbb{R}_{>0}$	$\mathbb{R}$
?	$\mathbb{C}$	

Last time

- Convolution $f * g : \mathbb{R} \to \mathbb{C}$ defined by

Sum $= x$

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - t)g(t)\,dt.$$



- Dirac kernel $K_t : \mathbb{R} \to \mathbb{R}$ ($t \in \mathbb{R}$, $t > 0$):
  - For all $t > 0$ and all $x \in \mathbb{R}$, $K_t(x) \geq 0$;
  - For all $t > 0$, $\displaystyle\int_{-\infty}^{\infty} K_t(x)\,dx = 1$; and
  - For fixed $\eta > 0$, we have $\displaystyle\lim_{t \to 0^+} \int_{|x| \geq \eta} K_t(x)\,dx = 0.$
- Thm: $\displaystyle\lim_{t \to 0^+} (f * K_t)(x) = f(x).$ $(f \in S(\mathbb{R}))$
- Example of a Dirac kernel: Gauss kernel

$$G_t(x) = \frac{1}{t} \exp\left(\frac{-\pi x^2}{t^2}\right).$$

$G_1 = e^{-\pi x^2}$

Operator notation for Fourier transform

Sometimes write $U(f)$ instead of $\hat{f}$. This goes with letting the transform variable be x instead of γ:

$$(U(f))(x) = \hat{f}(x) = \int_{-\infty}^{\infty} f(y) e^{-2\pi ixy}\, dy.$$

This makes it easier to think of the Fourier tranform as an operator U on the function space $\mathcal{S}(\mathbb{R})$. We will see momentarily that U sends $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$. Perhaps more importantly, in this notation, the inversion theorem boils down to proving that $U(U(f)) = s_{-1}(f)$, where the function $s_{-1}(f)$ is defined by $(s_{-1}(f))(x) = f(-x)$.

$\hat{\hat{f}}(x) = f(-x)$

Properties of Fourier transform

For $f \in \mathcal{S}(\mathbb{R})$:

Function (in x)	Fourier transform (in γ)
$f(x+a)$	$e^{2\pi i a\gamma}\hat{f}(\gamma)$
$e^{2\pi i a x}f(x)$	$\hat{f}(\gamma - a)$
$f(bx)$	$\frac{1}{b}\hat{f}\left(\frac{\gamma}{b}\right)$
$f(-x)$	$\hat{f}(-\gamma)$
$f'(x)$	$(2\pi i\gamma)f(\gamma)$
$(-2\pi i x)f(x)$	$\hat{f}'(\gamma)$

$\Leftarrow$ d/dx vs x

(cf. Laplace)

And crucially, for $f, g \in \mathcal{S}(\mathbb{R})$,

convolution $$\widehat{f * g}(\gamma) = \hat{f}(\gamma)\hat{g}(\gamma).$$ pointwise mult

Compare Fourier series: For $f, g \in C^0(S^1)$,

$$\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n).$$

Note!

$$\int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} f(u-a)\,du$$

$$= \int_{-\infty}^{\infty} f(x-a)\,dx$$

sub:

$u = x + a$

$du = dx$

$$\int_{-\infty}^{\infty} dx$$

trans. invt.

Proofs of properties of Fourier transform

Proofs use all of our old favorites: Substitution, parts, differentiation under the integral sign, Fubini.

As examples, we prove:

- If $g(x) = f(bx)$ $(b > 0)$, then $\hat{g}(\gamma) = \frac{1}{b}\hat{f}\left(\frac{\gamma}{b}\right)$.
- If $h(x) = f(-x)$, then $\hat{h}(\gamma) = \hat{f}(-\gamma)$.

Proof:

$$\hat{g}(\gamma) = \int_{-\infty}^{\infty} g(x) e^{-2\pi i \gamma x}\, dx$$

$$= \int_{-\infty}^{\infty} f(bx) e^{-2\pi i \gamma x}\, dx$$

$u = bx$

$du = b\,dx$

$x = \frac{u}{b}$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-2\pi i \gamma \left(\frac{u}{b}\right)} \left(\frac{1}{b}\right) du$$

As $x \to +\infty$, $u \to +\infty$.

" $x \to -\infty$, $u \to -\infty$

$$= \frac{1}{b} \int_{-\infty}^{\infty} f(u)\, e^{-2\pi i \left(\frac{\gamma}{b}\right) u}\, du$$

$$= \frac{1}{b} \hat{f}\left(\frac{\gamma}{b}\right).$$

$b = -1$?

Get $\quad -\int_{\infty}^{-\infty} \sim\sim = \int_{-\infty}^{\infty} \sim\sim = \hat{f}(-\gamma)$

The Fourier transform preserves $\mathcal{S}(\mathbb{R})$

Sketch:

Suppose $f \in \mathcal{S}(\mathbb{R})$. $f \in C^\infty$, $f^{(k)}$ decay rapidly

- By the table and induction, can show that for any $n, k \geq 0$, $\gamma^n \hat{f}^{(k)}(\gamma)$ is the Fourier transform of $\left(\frac{1}{2\pi i}\frac{d}{dx}\right)^n \left((-2\pi i x)^k f(x)\right)$. In particular, $\hat{f}^{(k)}(\gamma)$ exists for all $k \geq 0$.
- Fourier transform of any Schwartz function is bounded:

$g \in \mathcal{S}(\mathbb{R})$

$$|\hat{g}(\gamma)| = \left|\int_{-\infty}^{\infty} g(x) e^{-2\pi i \gamma x}\, dx\right| \leq \int_{-\infty}^{\infty} \left|g(x) e^{-2\pi i \gamma x}\right| dx = \int_{-\infty}^{\infty} |g(x)|\, dx = M$$

So $\gamma^n \hat{f}^{(k)}(\gamma)$ is bounded for all $n, k \geq 0$, i.e., $\hat{f} \in \mathcal{S}(\mathbb{R})$.

(ind of γ)

$\hat{f}^{(k)}(\gamma)$ is:

→ Take FT of f, get $\hat{f}(\gamma)$

→ Take $\left(\frac{d}{d\gamma}\right)^k$ of $\hat{f}$

So! If $f \in S(\mathbb{R})$, then $\hat{f} \in S(\mathbb{R})$.

(Ie: $\mathcal{U}: S(\mathbb{R}) \to S(\mathbb{R})$ is a lin op.)

The "Pass the hat" formula

Theorem

If $f, g \in \mathcal{S}(\mathbb{R})$, then $\int_{-\infty}^{\infty} \hat{f}(x)g(x)\,dx = \int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx.$

Proof:

$$\int_{-\infty}^{\infty} \hat{f}(x)\,g(x)\,dx$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y)\, e^{-2\pi i x y}\, dy \right) g(x)\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) f(y)\, e^{-2\pi i x y}\, dy\, dx \quad \boxed{\text{Fub}}$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x) f(y) e^{-2\pi i x y} \, dx \right) dy$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x) e^{-2\pi i x y} \, dx \right) f(y) \, dy$$

$$= \int_{-\infty}^{\infty} \hat{g}(y) f(y) \, dy = \int_{-\infty}^{\infty} f(x) \hat{g}(x) \, dx$$

The Gauss kernel

Theorem

The Fourier transform of $f(x) = e^{-\pi x^2}$ *is* $\hat{f}(\gamma) = e^{-\pi\gamma^2}$. *In other words,* f *is its own Fourier transform, or* $U(f) = f$.

More generally, for $t > 0$, *let* $G_t(x) = \frac{1}{t}\exp\left(\frac{-\pi x^2}{t^2}\right)$ *be the Gauss kernel. Then*

$$\hat{G}_t(\gamma) = e^{-\pi t^2\gamma^2}, \qquad U(U(G_t)) = \hat{\hat{G}}_t = G_t.$$

Proof: PS11.