Welcome to Math 131B

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 1.1–1.2, 2.1–2.2 Reading for Mon: 2.3–2.4.
- PS00 due Mon Aug 24; PS01 outline due Mon Aug 24; PS01 due Wed Aug 26.
- Problem session Fri Aug 21, 10:00–noon on Zoom.

Tour of the course website

The course website is:

`http://www.timhsu.net/courses/131b/`

Working in groups

In a minute, I'll send everyone into breakout rooms in groups of 3–4 to answer the following question:

What is one important event in your mathematical life?

In each breakout room:

- Learn **someone else's** name and important event. (I'll visit each room to help you organize cyclically.)
- Be ready to share that person's important event when we get back to the main room. (Take notes!)

Get ready to turn on your cameras and mics. (I'll pause the recording.)

Motivation 1: Two equations

Taylor e^x

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$x = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{\sin(2\pi n x)}{n\pi} \right)$$

$$= \frac{\sin(2\pi x)}{\pi} - \frac{\sin(4\pi x)}{2\pi} + \frac{\sin(6\pi x)}{3\pi} - \frac{\sin(8\pi x)}{4\pi} + \dots$$

What do those look like? (Maple)

not really f(x)=x, but segment between -.5 and .5, periodized

Q: What is really going on here?

Motivation 2: An equation and a non-equation

term by term differentiation: swap d/dx and inf sum

$$\frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{x^n}{n!}\right) = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = \sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x.$$

e^x

$$\frac{d}{dx}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{\sin(2\pi nx)}{n\pi}\right) \stackrel{?}{=} \sum_{n=1}^{\infty}(-1)^{n+1}\frac{d}{dx}\left(\frac{\sin(2\pi nx)}{n\pi}\right)$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}2\cos(2\pi nx).$$

Is it OK to swap d/dx and inf sum?

is this = 1?

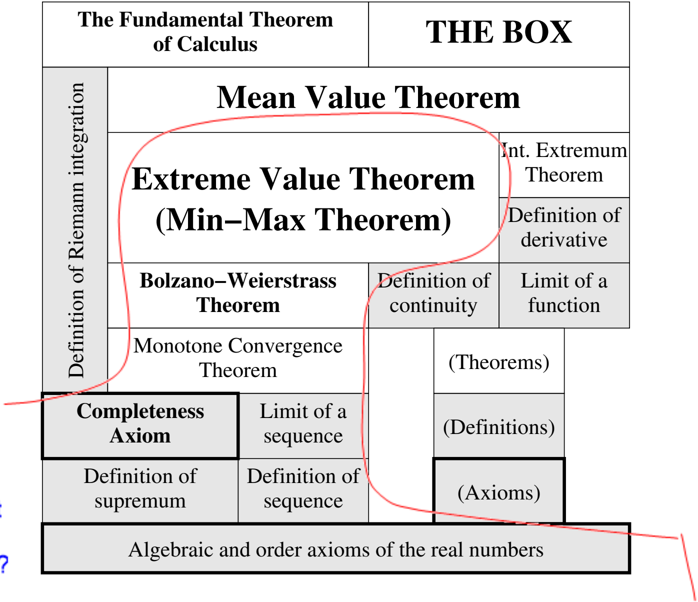
What does the second look like? (Maple)

PS01: Prove that this diverges for all rational x.

Motivation 3: Stringed instruments and harmonics

Watch:

https://www.youtube.com/watch?v=je1Epfxcg7s



I hope you at least got to this part in analysis?

But it's OK if you didn't! We'll start from scratch again.

Axioms for the real numbers: Field axioms

(A1) For all $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$. (+ associative)

(A2) For all $a, b \in \mathbb{R}$, $a + b = b + a$. (+ commutative)

(A3) There exists $0 \in \mathbb{R}$ such that for all $a \in \mathbb{R}$, $a + 0 = a$. (Zero)

(A4) For all $a \in \mathbb{R}$, there exists $(-a) \in \mathbb{R}$ such that $a + (-a) = 0$. (Negatives)

(M1) For all $a, b, c \in \mathbb{R}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. ($\cdot$ associative)

(M2) For all $a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$. ($\cdot$ commutative)

(M3) There exists $1 \in \mathbb{R}$, $1 \neq 0$, s.t. for all $a \in \mathbb{R}$, $a \cdot 1 = a$. (Unit)

(DL) For all $a, b, c \in \mathbb{R}$, $a \cdot (b + c) = a \cdot b + a \cdot c$. (Distributive)

(F1) For all $a \neq 0$ in $\mathbb{R}$, there exists $(1/a) \in \mathbb{R}$ such that $a \cdot (1/a) = 1$. (Reciprocals)

(F2) $1 \neq 0$. (Nontriviality)

Point: The real numbers have algebraic properties that you used in high school.

(A1)–(DL) defines a *ring*, e.g., $\mathbb{Z} =$ the integers.

Axioms for the real numbers: Order axioms

An *ordered field* satisfies axioms (A1)–(A4), (M1)–(M4), and (DL), and also has a relation $\leq$ such that:

(O1) For all $a, b \in \mathbb{R}$, either $a \leq b$ or $b \leq a$.

(O2) For all $a, b \in \mathbb{R}$, if $a \leq b$ and $b \leq a$, then $a = b$.

(O3) For all $a, b, c \in \mathbb{R}$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

(O4) For all $a, b, c \in \mathbb{R}$, if $a \leq b$, then $a + c \leq b + c$.

(O5) For all $a, b, c \in \mathbb{R}$, if $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Cor: If $c < 0$ and $a \leq b$, then $bc \leq ac$. (Flip!)

Also define:

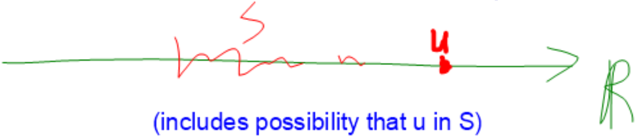
- $a < b$ means $a \leq b$ and $a \neq b$;
- $a \geq b$ means $b \leq a$;
- $a > b$ means $b < a$.

PS01

In a nutshell: Properties of $\leq, <, \geq, >$ are as you (maybe?) learned them in precalculus.

Both $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields; $\mathbb{C}$ is not orderable (i.e., no way to define $\leq$ consistent with the above).

Axioms for the real numbers: Order completeness



(includes possibility that u in S)

(OC) Every nonempty set of real numbers that has an upper bound also has a **least** upper bound (supremum).

It can be shown that the axioms (A1)–(A4), (M1)–(M3), (DL), (F1)–(F2), (O1)–(O5), and (OC) determine $\mathbb{R}$ completely; that is, any other object with the same properties must be essentially the same as $\mathbb{R}$.

For the rest of this course, we assume that there exists an object $\mathbb{R}$ that satisfies all of these axioms. All of our results ultimately rely only on these axioms.

S set of real numbers
To say that u is an upper bound for S means: u>=x for all x in S
The *least* upper bound of S is an upper bound that is <= all other upper bounds.

Density of the rationals

Theorem (Archimedean Prop)

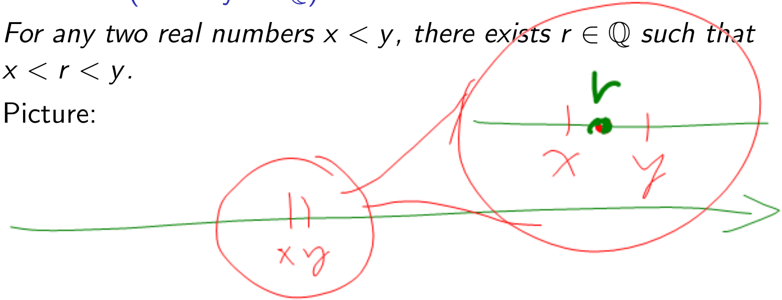
For any real number x, there is an integer $n > x$.

This plus some logic leads to:

Theorem (Density of $\mathbb{Q}$)

For any two real numbers $x < y$, there exists $r \in \mathbb{Q}$ such that $x < r < y$.

Picture:



We will use this specific theorem a little, and this picture **A LOT**.

Idea of density of Q: Rational are like dust that covers the real line

ACC and Sup Inequality

Theorem (Arbitrarily Close Criterion)

Suppose S is a nonempty subset of $\mathbb{R}$, and suppose u is an upper bound for S. Then the following are equivalent:

1. *For every $\epsilon > 0$, there exists some $s \in S$ such that $u - s < \epsilon$.*
2. $u = \sup S$.

Picture:

Lemma (Sup Inequality Lemma)

If S is a nonempty bounded subset of $\mathbb{R}$, then $\sup S \leq u$ if and only if u is an upper bound for S.

The complex numbers $\mathbb{C}$

Are polynomials in the variable i with real coefficients, with the relation $i^2 = -1$.
(Actually the fancy grownup definition of $\mathbb{C}$)

Picture:

Absolute value and conjugates

For $z = a + bi$ in $\mathbb{C}$, define:

$$|z| = \sqrt{a^2 + b^2}, \qquad \overline{a + bi} = a - bi$$

Lots of formulas that result from that and brute force; most frequently used include (for $z, w \in \mathbb{C}$):

$$|z|^2 = z\bar{z}, \qquad |zw| = |z|\,|w|$$