Sample final
Math 131B, Fall 2024

1. (12 points) Suppose $f(x) \in \mathcal{S}(\mathbf{R})$.

(a) For $f(x) \in \mathcal{S}(\mathbf{R})$, define the Fourier transform $\hat{f}(\gamma)$.

(b) What can you say about $\hat{\hat{f}}(x)$, the Fourier transform of the Fourier transform of f? No explanation necessary.

2. (14 points) Let $f : S^1 \to \mathbf{C}$ be given by

$$f(x) = \begin{cases} 0 & \text{for } -\dfrac{1}{2} \leq x < 0, \\ x & \text{for } 0 \leq x < \dfrac{1}{2}. \end{cases}$$

Calculate the Fourier coefficients $\hat{f}(n)$ $(n \in \mathbf{Z})$. Show all your work, and do not simplify your final answers.

In questions 3–8, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

3. (13 points) **TRUE/FALSE:** Let $f_n : [0,1] \to \mathbf{C}$ be a sequence of continuous functions that converges pointwise to some $f : [0,1] \to \mathbf{C}$. Then it must be the case that f is also continuous.

4. (13 points) **TRUE/FALSE:** Let $f : [2,5] \to \mathbf{C}$ be a differentiable function. Then it is possible that

$$\underline{\int_2^5} f(x)\,dx < \overline{\int_2^5} f(x)\,dx,$$

where the left-hand and right-hand sides of the inequality are the lower and upper Riemann integrals of f on $[2,5]$, respectively.

5. (13 points) **TRUE/FALSE:** Let $f : S^1 \to \mathbf{C}$ be a differentiable function such that f' is continuous on S^1. Then it must be the case that $\sum_{n \in \mathbf{Z}} \hat{f}(n) e_n(x)$ converges absolutely and uniformly to f.

6. (13 points) **TRUE/FALSE:** It is possible that there exists some $f \in L^2(S^1)$ such that the series $\sum_{n \in \mathbf{Z}} \left| \hat{f}(n) \right|^2$ diverges.

7. (13 points) **TRUE/FALSE:** Suppose $f, g \in \mathcal{S}(\mathbf{R})$ and $\hat{g}(\gamma) = 0$ for all $\gamma > 0$. Then it is possible that $\widehat{f * g}(13) = 7$.

8. (13 points) **TRUE/FALSE:** Let $f(x) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R = 7$. Then it must be the case that

$$f'(3) = \sum_{n=1}^{\infty} n a_n 3^{n-1}.$$

9. (16 points) **PROOF QUESTION.** Let $f : S^1 \to \mathbf{C}$ be given by

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x < \frac{1}{2}, \\ 0 & \text{for } \frac{1}{2} \leq x < 1. \end{cases}$$

It is a fact that

$$\hat{f}(0) = \frac{1}{2}, \qquad \hat{f}(n) = \begin{cases} 0 & \text{for } n \text{ even and } n \neq 0, \\ \frac{1}{\pi i n} & \text{for } n \text{ odd.} \end{cases}$$

(In other words, you may take the above as given and do not need to check it.

(a) Use an integral to compute $\|f\|^2$ (the L^2 norm of f).

(b) State *Parseval's identity*, which is the part of the Isomorphism Theorem for Fourier Series that gives a formula for $\|f\|^2$ as the sum of an infinite series.

(c) Prove that

$$\sum_{n>0,\ n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

10. (16 points) **PROOF QUESTION.** Suppose $f, K_n \in \mathcal{S}(\mathbf{R})$, $a \in \mathbf{R}$, and $K_n(x) \geq 0$ for all $x \in \mathbf{R}$. Let

$$b_n = \int_{-\infty}^{\infty} K_n(x)\, dx,$$

and suppose that $\lim_{n\to\infty} b_n = 0$.

(a) State the **definition** of what it means to say that $\lim_{n\to\infty} b_n = 0$ (i.e., the definition of the limit of a sequence).

(b) Prove that there exists some $M \in \mathbf{R}$ such that $|f(x+a) - f(x)| \leq M$ for all $x \in \mathbf{R}$.

(c) Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} (f(x+a) - f(x)) K_n(x)\, dx = 0.$$

11. (16 points) **PROOF QUESTION.** Suppose $a_n \in \mathbf{C}$ satisfy the condition that

$$|a_n| \leq \frac{1}{n^{3/2}} \qquad \text{for } n \neq 0.$$

Prove that the series $\sum_{n\in\mathbf{Z}} a_n e_n(x)$ converges absolutely and uniformly to a **continuous** function $g(x)$ on S^1. (In particular, make sure to justify the fact that $g(x)$ is continuous.)

12. (16 points) **PROOF QUESTION.** Let $f : \mathbf{R} \to \mathbf{C}$ be a function such that $f(7) = 5$ and the following condition holds:

> For every $\epsilon > 0$, there exists some $\delta(\epsilon) > 0$ such that if $|x - 7| < \delta(\epsilon)$, then $|f(x) - 5| < \epsilon$.

Prove that

$$\lim_{n\to\infty} f\left(7 + \frac{1}{n}\right) = 5.$$

13. (16 points) **PROOF QUESTION.** Suppose $c_n \in \mathbf{C}$, and suppose that the series

$$f(x) = \sum_{n\in\mathbf{Z}} c_n e_n(x)$$

converges uniformly. Prove that for $k \in \mathbf{Z}$, we have that

$$\hat{f}(k) = c_k.$$

In particular, carefully **JUSTIFY** your work whenever you pull out/push in an infinite sum or limit.

14. (16 points) **PROOF QUESTION.** Let $h \in \mathcal{S}(\mathbf{R})$ be given by $h(x) = xe^{-\pi x^2}$.

(a) If $g(x) = e^{-\pi x^2}$, what is $\hat{g}(\gamma)$? No explanation necessary.

(b) Prove that

$$\hat{h}(\gamma) = (-i)h(\gamma).$$

Note: There are multiple ways to prove this using methods from this class, but in particular, if you use integration by parts, make sure to justify any evaluations of definite integrals that you do.