Sample exam 3
Math 131B, Fall 2024

1. (10 points) For each of the following classes of functions f, state the best results that we have seen about the convergence of the Fourier series of f. In particular, when you state a convergence result, be careful to specify the **type** of convergence in question.

(a) $f \in L^2(S^1)$.

(b) $f \in C^0(S^1)$. (Note that the best result that applies to *all* continuous f is not actually about the Fourier series of f, per se.)

(c) $f \in C^1(S^1)$.

In questions 2–4, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (12 points) **TRUE/FALSE:** Let $\{u_n \mid n \in \mathbf{N}\}$ be a set of nonzero orthogonal vectors in a Hilbert space $\mathcal{H}$ with the property that:

For any $f \in \mathcal{H}$, if $\langle f, u_n \rangle = 0$ for all $n \in \mathbf{N}$, then $f = 0$.

Then it is possible that there exists some $f \in \mathcal{H}$ such that the generalized Fourier series $\sum_{n=1}^{\infty} \hat{f}(n) u_n$ does *not* converge to f in the inner product metric.

3. (12 points) **TRUE/FALSE:** Let $f, g \in C^0(S^1)$ be functions such that $\hat{g}(2k) = 0$ for all $k \in \mathbf{Z}$. Then it is possible that $\widehat{(f * g)}(6) = 3 + 2i$.

4. (12 points) **TRUE/FALSE:** For any $f \in L^2(S^1)$ and $c_{-8}, \dots, c_0, \dots, c_8 \in \mathbf{C}$, it must be the case that

$$\|f - f_8\| \leq \left\| f - \sum_{n=-8}^{8} c_n e_n \right\|,$$

where f_8 is the Fourier polynomial of f of degree 8.

5. (13 points) **PROOF QUESTION.** Let $f : S^1 \to \mathbf{C}$ be continuous, and recall that $e_n(x) = e^{2\pi i n x}$. Let

$$G_N(x) = \sum_{k=0}^{N} e_{2k}(x).$$

Use the **definition** of convolution to prove that

$$(G_N * f)(x) = \sum_{k=0}^{N} \hat{f}(2k) e_{2k}(x).$$

6. (13 points) **PROOF QUESTION.** Let $\mathcal{H}$ be a Hilbert space, let $\{u_n \mid n \in \mathbf{N}\}$ be an orthogonal set, and suppose that $c_n \in \mathbf{C}$ are such that $f = \sum_{n=1}^{\infty} c_n u_n$ converges in the IP metric in $\mathcal{H}$.

(a) Define $f = \sum_{n=1}^{\infty} c_n u_n$ as a limit of finite sums.

(b) Prove that for fixed $k \geq 1$, we have that

$$\langle f, u_k \rangle = c_k \|u_k\|^2 .$$

Briefly **JUSTIFY** each step, and in particular, each time you pull out a limit or an infinite sum, make sure to **JUSTIFY** that operation carefully.

7. (14 points) **PROOF QUESTION.** Let $f : S^1 \to \mathbf{R}$ be integrable, with

$$\int_{-1/2}^{1/2} |f(t)| \, dt = 3,$$

and let $h(x,t)$ be a real-valued function in two variables $x, t \in S^1$ such that $h(x,t)$ is continuous in each variable. Suppose h also satisfies the following condition:

Condition on $h(x,t)$. For any $\epsilon_1 > 0$, there exists some $\delta_1(\epsilon_1) > 0$ such that if $|x| < \delta_1(\epsilon_1)$, then $|h(x,t) - h(0,t)| < \epsilon_1$.

Let

$$g(x) = \int_{-1/2}^{1/2} h(x,t) f(t) \, dt.$$

Prove that for every $\epsilon > 0$, there exists some $\delta > 0$ such that if $|x| < \delta$, then $|g(x) - g(0)| < \epsilon$.

8. (14 points) **PROOF QUESTION.** Let $\{e_n \mid n \in \mathbf{Z}\}$ be the usual basis for $L^2(S^1)$ and suppose $a_n \in \mathbf{C}$.

(a) State the Hilbert Space Absolute Convergence Theorem for $\sum_{n \in \mathbf{Z}} a_n e_n$.

(b) Now suppose that for all $n \in \mathbf{Z}$,

$$|a_n| \leq \frac{1}{|n|^{3/4} + 5}.$$

Prove that $\sum_{n \in \mathbf{Z}} a_n e_n$ converges (in the inner product metric) to some $f \in L^2(S^1)$.