Sample exam 2
Math 131B, Fall 2024

1. (12 points) Suppose V is a function space, $\langle \cdot, \cdot \rangle : V \times V \to \mathbf{C}$ is an inner product, and f, g, h are nonzero elements of V such that

$$\langle f, f \rangle = 4, \qquad \langle f, g \rangle = -3i, \qquad \langle g, g \rangle = 2.$$

(a) What can you say about $\langle h, h \rangle$? No explanation necessary, but be precise.

(b) Compute the value of $\langle f - g, 2f + (5i)g \rangle$. You do not need to simplify your final numerical answer, but show all your work.

2. (14 points) Let $f : S^1 \to \mathbf{C}$ be given by

$$f(x) = \begin{cases} -x & \text{for } -\frac{1}{2} \le x \le -\frac{1}{4}, \\ 0 & \text{for } -\frac{1}{4} < x < \frac{1}{2}. \end{cases}$$

Calculate the Fourier coefficients $\hat{f}(n)$ $(n \in \mathbf{Z})$. Show all your work, and do not simplify your final answers.

In questions 3–5, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

3. (12 points) **TRUE/FALSE:** If $n, k \in \mathbf{Z}$ and $n \neq k$, then $\displaystyle\int_0^1 e_n(x)\overline{e_k(x)}\,dx = 0$.

4. (12 points) **TRUE/FALSE:** If $f_n, f : [0,1] \to \mathbf{C}$ are functions such that each f_n is differentiable and f_n converges pointwise to f, then it must be the case that f is continuous.

5. (12 points) **TRUE/FALSE:** It is possible that there exists some $f \in C^2(S^1)$ such that $\left|\hat{f}(n)\right| = \dfrac{1}{n}$ for all $n \geq 1$.

6. (12 points) **PROOF QUESTION.** Let $a_n, b_k \in \mathbf{C}$ be constants, and suppose that the series

$$f(x) = \sum_{n=1}^{\infty} a_n e_n(x), \qquad g(x) = \sum_{k=1}^{\infty} b_k e_k(x),$$

converge uniformly on S^1. Prove that

$$\int_0^1 f(x)\overline{g(x)}\,dx = \sum_{n=1}^{\infty} a_n \overline{b_n}.$$

Make sure to indicate where you use uniform convergence.

7. (12 points) **PROOF QUESTION.** Note that in this problem, you are not allowed to assume the existence of, or use the properties of, log functions of any kind.

Let $C \in \mathbf{R}$ be constant, and let $L : (0, \infty) \to \mathbf{R}$ be a differentiable function such that $L(1) = 0$ and for all $x > 0$, we have

$$L'(x) = \frac{C}{x}.$$

Let $a > 0$ be constant, and let $f(x) = L(ax) - L(x)$.

(a) Compute $f'(x)$. (Suggestion: Remember the chain rule.)

(b) What can we conclude about f from part (a)?

(c) Prove that for all $x \in \mathbf{R}$, we have that

$$L(ax) - L(x) = L(a).$$

8. (14 points) **PROOF QUESTION.** Let f be an element of $C^0(S^1)$.

(a) For $g_n \in C^0(S^1)$, describe what the Weierstrass M-test tells you when applied to the function series $\sum_{n \in \mathbf{Z}} g_n(z)$. Be clear about assumptions and conclusions.

(b) Define the Fourier series of f.

(c) Suppose now that for all $n \neq 0$, we have that

$$\left|\hat{f}(n)\right| \leq \frac{3}{|n|^{5/4}}.$$

Prove that the Fourier series of f converges absolutely and uniformly on S^1 to some function g.