Sample exam 1
Math 131B, Fall 2024

1. (19 points) Let $v : [a, b] \to \mathbf{R}$ be a bounded nonnegative function. Recall that a *partition* P of $[a, b]$ is a subset $\{x_0, \ldots, x_n\} \subset [a, b]$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Let $(\Delta x)_i = x_i - x_{i-1}$ and let $\mathcal{P}$ be the set of all partitions of $[a, b]$.

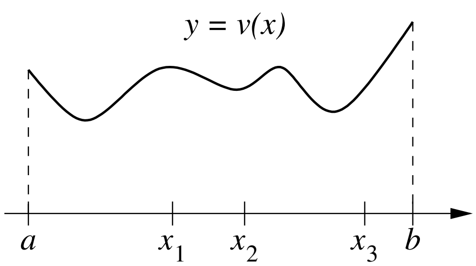
(a) Define $m(v; P, i)$.

(b) Draw a picture of what $m(v; P, 3)$ represents for the $v(x)$ and the subinterval $[x_2, x_3]$ shown below.

(c) For a partition $P \in \mathcal{P}$, define $L(v; P)$.

(d) Define the lower Riemann integral $\underline{\int_a^b} v(x)\, dx$.

You may find it helpful to draw more pictures to illustrate the above answers.



In questions 2–4, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (13 points) **TRUE/FALSE:** Let S be a nonempty bounded subset of $\mathbf{R}$ such that for all $s \in S$, we have $s \leq 11$. Then it is possible that $\sup S > 11$.

3. (13 points) **TRUE/FALSE:** Let $f : [2, 7] \to \mathbf{R}$ be a function such that for every $a \in [2, 7]$, the limit $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. Then it must be the case that

$$\underline{\int_2^7} f(x)\, dx = \overline{\int_2^7} f(x)\, dx.$$

4. (13 points) **TRUE/FALSE:** Let $f : \mathbf{C} \to \mathbf{C}$ and $g : \mathbf{C} \to \mathbf{C}$ be functions such that:

- For any $a \in \mathbf{C}$, if $\lim_{n \to \infty} x_n = a$, then $\lim_{n \to \infty} f(x_n) = f(a)$; and
- For any $b \in \mathbf{C}$ and any $\epsilon > 0$, there exists some $\delta(\epsilon) > 0$ such that if $|y - b| < \delta(\epsilon)$, then $|g(y) - g(b)| < \epsilon$.

Then it is possible that $(g \circ f)(x)$ is not continuous at $x = 4$.

5. (14 points) **PROOF QUESTION.** Let X be a closed and bounded subset of $\mathbf{C}$, and let $f : X \to \mathbf{R}$ be a continuous function.

(a) State the Bolzano-Weierstrass Theorem for sequences in X.

(b) Suppose that z_n is a sequence in X such that $\lim_{n\to\infty} f(z_n) = 13$. Prove that there exists some $a \in X$ such that $f(a) = 13$. (If you do not see how to prove the full statement, for partial credit, you may add the extra assumption that z_n is a convergent sequence.)

6. (14 points) **PROOF QUESTION.** Let X be a metric space and let L be a point in X.

(a) Let x_n be a sequence in X. Define what it means to say that $\lim_{n\to\infty} x_n = L$. (Note that x_n and L may not be numbers, just elements of the metric space X.)

(b) In the following proof, use only the definition of $\lim_{n\to\infty} x_n = L$ and not any results you may have proven on the homework. (I.e., the point of this proof is basically to redo a problem from the homework.)

For all n, suppose we know that

$$d(x_n, L) \leq \frac{1}{\sqrt{n}}.$$

Prove that $\lim_{n\to\infty} x_n = L$.

7. (14 points) **PROOF QUESTION.** Suppose that X and Y are nonempty subsets of $\mathbf{C}$, and suppose that $f : X \to Y$ and $g : Y \to \mathbf{C}$ are functions such that:

- f is continuous on X, $f(a) = b$, and $f(x) \neq b$ whenever $x \in X$ and $x \neq a$.
- b is a limit point of Y and $\lim_{y\to b} g(y) = c$.

Prove that $\lim_{x\to a} g(f(x)) = c$. (For partial credit, state either the sequential or ϵ-δ definition of $\lim_{y\to b} g(y) = c$.)