Sample Final Exam
Math 131B, Fall 2023

1. (14 points) Let $f : S^1 \to \mathbf{C}$ be given by

$$f(x) = \begin{cases} 3 & \text{for } -\frac{1}{2} \leq x < 0, \\ 0 & \text{for } 0 \leq x < \frac{1}{2}. \end{cases}$$

Calculate the Fourier coefficients $\hat{f}(n)$ ($n \in \mathbf{Z}$). Show all your work, and do not simplify your final answers.

2. (16 points) Let f be in the Schwartz space $\mathcal{S}(\mathbf{R})$.

(a) Define the Fourier Transform $\hat{f}(\gamma)$ of f.

(b) What does the Inversion Theorem for the Fourier Transform say about $\hat{\hat{f}}$ (the Fourier transform of the Fourier transform of f)? Briefly state the result.

3. (12 points) Let X and Y be metric spaces and let $f : X \to Y$ be a function. State the ϵ-δ definition of what it means for f to be continuous at $a \in X$.

In questions 4–9, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

4. (13 points) **TRUE/FALSE:** It is possible that there exist $f, g \in L^2(S^1)$ such that $\hat{f}(5) = 2\pi$, $\hat{g}(5) = -13i$, and $(\widehat{f * g})(5) = 33$.

5. (13 points) **TRUE/FALSE:** For $n \geq 1$, let $g_n : [0,1] \to \mathbf{C}$ be continuous, and suppose that the sequence $g_n(x)$ converges pointwise to some $g : [0,1] \to \mathbf{C}$. Then it must be the case that g is continuous.

6. (13 points) **TRUE/FALSE:** For $h \in L^2(S^1)$, it must be the case that

$$\lim_{N \to \infty} \left\| h(x) - \sum_{n=-N}^{N} \hat{h}(n) e_n(x) \right\| = 0,$$

where the norm in the limit is the L^2 (inner product) norm.

7. (13 points) **TRUE/FALSE:** Let $f : \mathbf{R} \to \mathbf{C}$ be a continuous function with $f(6) = -13$. Then it is possible that

$$\lim_{n \to \infty} f\left(6 - \frac{1}{n^2}\right) = 12.$$

8. (13 points) **TRUE/FALSE:** For any $x \in \mathbf{R}$, it must be the case that $\left|e^{ix}\right| = 1$.

9. (13 points) **TRUE/FALSE:** It is possible that there exists some $g \in L^2(S^1)$ and some $a_n \in \mathbf{C}$ such that

$$\left\| g(x) - \sum_{n=-10}^{10} \hat{g}(n) e_n(x) \right\| = 5, \qquad \left\| g(x) - \sum_{n=-10}^{10} a_n e_n(x) \right\| = 4.$$

10. (16 points) **PROOF QUESTION.** Let $\{e_n \mid n \in \mathbf{Z}\}$ be the usual orthonormal basis for $L^2(S^1)$.

(a) State the Hilbert Space Absolute Convergence Theorem for $\sum_{n \in \mathbf{Z}} a_n e_n$.

(b) Suppose $a_n \in \mathbf{C}$ is a two-sided sequence such that

$$|a_n| \le \frac{1}{|n|^{3/2}} \qquad \text{for all } n \neq 0.$$

Prove that $\sum_{n \in \mathbf{Z}} a_n e_n(x)$ converges to some $f \in L^2(S^1)$.

11. (16 points) **PROOF QUESTION.** Let $\mathcal{H}$ be a Hilbert space, and let $\{u_n \mid n \ge 1\}$ be an ortho**normal** set in $\mathcal{H}$. Suppose also that for some $a_n, c_k \in \mathbf{C}$, we have that

$$f = \sum_{n=1}^{\infty} a_n u_n, \qquad g = \sum_{k=1}^{\infty} c_k u_k,$$

where the convergence of each series is in the inner product metric.

Prove carefully that $\langle f, g \rangle = \sum_{n=1}^{\infty} a_n \overline{c_n}$. In particular, be clear about when you are using the definition of an inner product and when you are using continuity.

12. (16 points) **PROOF QUESTION.** Suppose $f, g_n \in \mathcal{S}(\mathbf{R})$ are functions such that

$$g(x) = \sum_{n=1}^{\infty} f_n(x),$$

where convergence is uniform on $\mathbf{R}$. Prove that for any $\gamma \in \mathbf{R}$, we have that

$$\hat{g}(\gamma) = \sum_{n=1}^{\infty} \hat{f}_n(\gamma).$$

Make sure you state clearly where you use uniform convergence.

13. (16 points) **PROOF QUESTION.** Prove that

$$\sum_{n=0}^{\infty} \left(\frac{x^n}{4^n} \right) e^{\pi n^2 i x}$$

converges absolutely and uniformly to a continuous function on $[-3, 3]$. (Suggestion: How do you express "$x \in [-3, 3]$" in terms of $|x|$?)

14. (16 points) **PROOF QUESTION.** Suppose $f \in \mathcal{S}(\mathbf{R})$, which means that all derivatives of f are in $\mathcal{S}(\mathbf{R})$ (i.e., you may take that as given). Let $g(x) = f''(x)$ (the second derivative of $f(x)$).

Use the definition of Fourier transform and integration by parts to prove that for $\gamma \in \mathbf{R}$,

$$\hat{g}(\gamma) = -4\pi^2\gamma^2\hat{f}(\gamma).$$

(For full credit, do **not** just cite the results about $\widehat{\frac{df}{dx}}$ proven in Chapter 12; instead, imitate the *proof* of those results, as seen in your homework.)