Sample exam 3
Math 131B, Fall 2023

1. (10 points) Let E be a subset of $\mathbf{R}$. Define what it means for E to have measure zero.

In questions 2–4, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (12 points) **TRUE/FALSE:** Suppose $f \in C^1(S^1)$. Then it is possible that for some $x \in S^1$, the series $\sum_{n\in\mathbf{Z}} \hat{f}(n)e_n(x)$ diverges.

3. (12 points) **TRUE/FALSE:** Suppose $g : S^1 \to \mathbf{C}$ is continuous and $K_N(x)$ is a Dirac kernel. Then it must be the case that $g * K_N$ converges uniformly to g on S^1 as $N \to \infty$.

4. (12 points) **TRUE/FALSE:** It is possible that there exist some $f \in L^2(S^1)$ and some $c_{-6}, \ldots, c_0, \ldots, c_6 \in \mathbf{C}$ such that

$$\|f - f_6\| = 1 \qquad \text{and} \qquad \left\| f - \sum_{n=-6}^{6} c_n e_n \right\| = 1/2,$$

where f_6 is the Fourier polynomial of f of degree 6.

5. (13 points) **PROOF QUESTION.** Let $\mathcal{H}$ be a Hilbert space and fix $g \in \mathcal{H}$.

(a) Define what it means for $T : \mathcal{H} \to \mathcal{H}$ to be continuous at g.

(b) Now let $T : \mathcal{H} \to \mathcal{H}$ be a function such that for all $f, g \in \mathcal{H}$ and $a, b \in \mathbf{C}$, we have that

$$T(af + bg) = aT(f) + bT(g), \qquad \|T(f)\| = \|f\|.$$

Prove that T is continuous at g.

6. (13 points) **PROOF QUESTION.** Let $f : S^1 \to \mathbf{C}$ be given by

$$f(x) = x \qquad \text{for } 0 \leq x < 1.$$

(a) State Parseval's Identity about $\|f\|^2$ for $f \in L^2(S^1)$.

(b) Prove that

$$\sum_{n\in\mathbf{Z}} \left|\hat{f}(n)\right|^2 = \frac{1}{3}.$$

7. (13 points) **PROOF QUESTION.** Suppose $f, g \in C^0(S^1)$ are functions such that

$$\hat{f}(n) = \begin{cases} \dfrac{1}{n^3} & n \text{ odd}, \\ 0 & n \text{ even}, \end{cases} \qquad \hat{g}(n) = \begin{cases} 0 & n \text{ odd}, \\ \dfrac{1}{n^2+1} & n \text{ even}. \end{cases}$$

Prove that $(f * g)(x) = 0$.

8. (15 points) **PROOF QUESTION.** Let $\{e_n \mid n \in \mathbf{Z}\}$ be the usual basis for $L^2(S^1)$ and suppose $a_n \in \mathbf{C}$.

(a) State the Hilbert Space Absolute Convergence Theorem for $\sum_{n \in \mathbf{Z}} a_n e_n$.

(b) Now suppose that for all $n \in \mathbf{Z}$,

$$|a_n| \leq \frac{13}{|n| + 1}.$$

Prove that $\sum_{n \in \mathbf{Z}} a_n e_n$ converges (in the inner product metric) to some $f \in L^2(S^1)$.