Sample exam 2
Math 131B, Fall 2023

1. (12 points) Define what it means for a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbf{C}$ to be an inner product, i.e., state the three axioms of an inner product (linear in the first variable, Hermitian, positive definite).

2. (14 points) Let $f : S^1 \to \mathbf{C}$ be given by

$$f(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq \frac{1}{4}, \\ 3x & \text{for } \frac{1}{4} < x < 1. \end{cases}$$

Calculate the Fourier coefficients $\hat{f}(n)$ ($n \in \mathbf{Z}$). Show all your work, and do not simplify your final answers.

In questions 3–5, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

3. (12 points) **TRUE/FALSE:** Let $p(x) = \sum_{n=-N}^{N} c_n e_n(x)$ for some $c_n \in \mathbf{C}$. It is possible that $\langle p, e_k \rangle = 0$ for all $k \in \mathbf{Z}$ and $p(x) \neq 0$.

4. (12 points) **TRUE/FALSE:** For any $n, k \in \mathbf{Z}$, we have that $\int_0^1 e_n(x)\overline{e_k(x)}\, dx = 0$.

5. (12 points) **TRUE/FALSE:** Let $f_n : [a, b] \to \mathbf{C}$ be a sequence of integrable functions that converges uniformly to some integrable function $f : [a, b] \to \mathbf{C}$. It must be the case that for any $\epsilon > 0$, there exists some $N(\epsilon)$ such that if $n > N(\epsilon)$, then

$$\left| \int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx \right| < \epsilon.$$

6. (12 points) **PROOF QUESTION.** For $n \geq 1$, define $f_n : [0, 1] \to \mathbf{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x = 0, \\ n^2 & \text{if } 0 < x < \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} \leq x \leq 1, \end{cases}$$

and define $f : [0, 1] \to \mathbf{R}$ by $f(x) = 0$.

(a) Use the definition of the limit of a sequence to prove that if x is a fixed real number such that $0 < x \leq 1$, then $\lim_{n \to \infty} f_n(x) = f(x)$.

(b) Explain why $\int_0^1 f_n(x)\, dx$ does not converge to $\int_0^1 f(x)\, dx$. (You may prefer to draw pictures to compute the integrals in question.)

7. (12 points) **PROOF QUESTION.** Let $f, g : \mathbf{R} \to \mathbf{R}$ be differentiable functions such that for $x \in \mathbf{R}$,

$$f'(x) = g(x), \qquad g'(x) = -f(x),$$
$$f(7) = 2, \qquad g(7) = 3.$$

Let $h(x) = f(x)^2 + g(x)^2$.

(a) Prove that for all $x \in \mathbf{R}$, we have that $h'(x) = 0$. (Suggestion: Remember the chain rule.)

(b) What can we conclude about h from the fact that $h'(x) = 0$?

(c) Prove that for all $x \in \mathbf{R}$, we have that

$$f(x)^2 + g(x)^2 = 13.$$

8. (14 points) **PROOF QUESTION.** Let X be a subset of $\mathbf{C}$, and let $f_n : X \to \mathbf{C}$ be a sequence of **continuous** functions.

(a) Describe what the Weierstrass M-test tells you when applied to the function series $\sum_{n=1}^{\infty} f_n(z)$. Be clear about assumptions and conclusions.

(b) Suppose now that for $n \geq 1$ and $z \in X$, we have that

$$|f_n(z)| \leq \frac{7}{n^{3/2}}.$$

Prove that the series

$$\sum_{n=1}^{\infty} f_n(z)$$

converges absolutely and uniformly on X to a **continuous** function.