Sample exam 1
Math 131B, Fall 2023

1. (16 points) Let $v : [a, b] \to \mathbf{R}$ be a bounded nonnegative function. Recall that a *partition* P of $[a, b]$ is a finite subset $\{x_0, \dots, x_n\} \subset [a, b]$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, let $(\Delta x)_i = x_i - x_{i-1}$, and define

$$M(v; P, i) = \sup \{v(x) \mid x \in [x_{i-1}, x_i]\}, \qquad m(v; P, i) = \inf \{v(x) \mid x \in [x_{i-1}, x_i]\}.$$

Let $\mathcal{P}$ be the set of all partitions of $[a, b]$.

(a) For a partition $P \in \mathcal{P}$, define $U(v; P)$.

(b) Define the upper Riemann integral $\overline{\int_a^b} v(x)\, dx$.

(c) In part (b), you used either a sup or an inf in your definition. Briefly **EXPLAIN** why the sup or inf in this definition gives a "best possible estimate" of the area under the curve $v(x)$ of a particular type.

You may find it helpful to draw a picture or pictures to illustrate the above answers.

In questions 2–4, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (13 points) **TRUE/FALSE:** Let $v : [1, 5] \to \mathbf{R}$ be a function such that for every $a \in [1, 5]$, the limit $\lim_{x \to a} \dfrac{v(x) - v(a)}{x - a}$ exists. Then it is possible that for every partition P of $[1, 5]$, we have that such that $U(v; P) - L(v; P) \geq \dfrac{1}{3}$.

3. (13 points) **TRUE/FALSE:** Let a_n be a sequence in $\mathbf{C}$ such that for every $\epsilon > 0$, there exists some $N(\epsilon)$ such that for every $n, k > N(\epsilon)$, we have that $|a_n - a_k| < \epsilon$. Then it must be the case that $\lim_{n \to \infty} a_n = L$ for some $L \in \mathbf{C}$.

4. (13 points) **TRUE/FALSE:** Let S be a nonempty subset of $\mathbf{R}$ such that for all $x \in S$, $x \geq 3$. Then it must be the case that $\inf S = 3$.

5. (15 points) **PROOF QUESTION.** Let X be a closed and bounded subset of $\mathbf{C}$, and let $f : X \to \mathbf{R}$ be a continuous function.

(a) State the Bolzano-Weierstrass Theorem for sequences in X.

(b) State the sequential definition of what it means for f to be continuous at $a \in X$.

(c) Suppose that x_n is a sequence in X such that $\lim_{n \to \infty} f(x_n) = +\infty$. Prove that there exists some $a \in X$ such that $f(a) = +\infty$, and therefore, by contradiction, no such sequence x_n can exist. (If you do not see how to prove the full statement, for partial credit, you may add the extra assumption that x_n is a convergent sequence.)

6. (15 points) **PROOF QUESTION.** Let $f : [2, 9] \to \mathbf{C}$ be a function.

(a) State the ϵ-δ definition of what it means for f be continuous at $a = 6$.

(b) Now suppose that, for any $x, y \in [2, 9]$, we have

$$|f(x) - f(y)| \leq 37 |x - y| .$$

Prove that f is continuous at $a = 6$.

7. (15 points) **PROOF QUESTION.** Let X be a metric space and let L be a point in X.

(a) Let x_n be a sequence in X. Define what it means to say that $\lim_{n\to\infty} x_n = L$. (Note that x_n and L may not be numbers, just elements of the metric space X.)

(b) For all n, let

$$a_n = 3d(x_n, L).$$

Prove that if $\lim_{n\to\infty} x_n = L$, then $\lim_{n\to\infty} a_n = 0$.