Sample Final Exam
Math 131B, Fall 2021

1. (14 points) Let f be in $L^2(S^1)$, and N be a positive integer.

(a) Define $f_N(x)$, the Nth Fourier polynomial of f.

(b) Define what it means for $p(x)$ to be a trigonometric polynomial of degree N.

(c) State the *Best Approximation Theorem.* (I.e., what is the most notable property of the Nth Fourier polynomial of f?)

2. (14 points) Suppose $f, g \in C^0(S^1)$.

(a) Define the convolution $(f * g)(x)$.

(b) What is the most notable property of the Fourier coefficients of $f*g$? State the formula precisely.

3. (14 points) Calculate the Fourier coefficients $\hat{f}(n)$ of the function $f : S^1 \to \mathbf{C}$ given for $-\frac{1}{2} \le x \le \frac{1}{2}$ by

$$f(x) = \begin{cases} 1 & \text{if } -\frac{1}{4} \le x \le \frac{1}{4}, \\ 0 & \text{otherwise.} \end{cases}$$

Show all your work, and do not simplify your final answer.

For questions 4–7, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

4. (13 points) **TRUE/FALSE.** It is possible that there exists a sequence a_n in $\mathbf{R}$ and a continuous function $f : \mathbf{R} \to \mathbf{C}$ such that $\lim_{n\to\infty} a_n = 5$, $f(5) = 13$, and $\lim_{n\to\infty} f(a_n) = 7$.

5. (13 points) **TRUE/FALSE.** If $f : [1, 5] \to \mathbf{C}$ is a Riemann integrable function, then it must be the case that f is continuous.

6. (13 points) **TRUE/FALSE.** Suppose $f \in L^2(S^1)$, and let f_N be the Nth Fourier polynomial of f. Then it must be the case that $\lim_{N\to\infty} \|f - f_N\| = 0$.

7. (13 points) **TRUE/FALSE.** Let f_n be a sequence of continuous functions on $[0, 1]$, and suppose that $f : [0, 1] \to \mathbf{C}$ is a function such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [0, 1]$. Then it must be the case that f is continuous.

8. (16 points) **PROOF QUESTION.** Suppose $f \in C^3(S^1)$. Prove that

$$\sum_{n\in\mathbf{Z}} (2\pi n)\hat{f}(n)$$

converges absolutely. (Suggestion: You may find the Extra Derivative Lemma to be helpful, though it is not necessary for this problem.)

9. (16 points) **PROOF QUESTION.** For $k = 0$ and $k = 1$, define $f_k : S^1 \to \mathbf{C}$ by

$$f_k(x) = \sum_{n \neq 0} \left(\frac{(2\pi i n)^k}{n^3} \right) e_n(x).$$

(a) Prove that if either $k = 0$ or $k = 1$, then $f_k(x)$ converges absolutely and uniformly on S^1.

(b) Prove that $f_0(x)$ is differentiable and $f_0'(x) = f_1(x)$. Be precise about the hypotheses you need to make term-by-term differentiation work.

10. (16 points) **PROOF QUESTION.** Prove that for $f \in C^0(S^1)$ and $n \in \mathbf{Z}$, we have that

$$(e_n * f)(x) = \hat{f}(n) e_n(x).$$

11. (16 points) **PROOF QUESTION.** For $f \in L^2(S^1)$, define $u : S^1 \times (0, +\infty) \to \mathbf{C}$ and $h : (0, +\infty) \to \mathbf{C}$ by

$$u(x,t) = \sum_{n \in \mathbf{Z}} \left(\frac{1}{t^2 + 1} \right) \hat{f}(n) e_n(x),$$
$$h(t) = \|u(x,t)\|^2,$$

where the norm $\|u(x,t)\|$ is computed in $L^2_x(S^1)$, holding t constant.

(a) Use Parseval to prove that $h(t)$ is equal to a function series in t.

(b) Prove that $h(t)$ converges absolutely and uniformly to a continuous function.

12. (15 points) Let $\mathcal{H}$ be a Hilbert space, and let $\{u_n \mid n \in \mathbf{N}\}$ be a set of nonzero vectors in $\mathcal{H}$.

(a) Define what it means for $\{u_n \mid n \in \mathbf{N}\}$ to be an orthogonal set.

(b) Define what it means for $\{u_n \mid n \in \mathbf{N}\}$ to be an orthogonal basis.

13. (15 points)

(a) For f in the Schwartz space $\mathcal{S}(\mathbf{R})$, define the Fourier transform $\hat{f}(u)$ of f.

(b) State the Fourier inversion theorem in the case of $f \in \mathcal{S}(\mathbf{R})$. (In other words, what do you need to do to $\hat{f}$ to recover f?)

14. (15 points) **PROOF QUESTION.** Suppose $f \in \mathcal{S}(\mathbf{R})$, and let $g(x) = f(-x)$. Prove that for $\gamma \in \mathbf{R}$, $\hat{g}(\gamma) = \hat{f}(-\gamma)$.